Mathematical Tables and other Aids to Computation

A Quarterly Journal edited on behalf of the
Committee on Mathematical Tables
and Other Aids to Computation
by

RAYMOND CLARE ARCHIBALD
DERRICK HENRY LEHMER

WITH THE COÖPERATION OF

LESLIE JOHN COMRIE
SOLOMON ACHILLOVICH JOFFE

I · Number 6 · April 1944

Published by

THE NATIONAL RESEARCH COUNCIL

NATIONAL RESEARCH COUNCIL

Division of Physical Sciences

COMMITTEE ON MATHEMATICAL TABLES AND OTHER AIDS TO COMPUTATION

Published quarterly in January, April, July and October by the National Research Council, Prince and Lemon Sts., Lancaster, Pa., and Washington, D. C.

All contributions intended for publication in *Mathematical Tables and Other Aids to Computation*, and all Books for review, should be addressed to Professor R. C. Archibald, Brown University, Providence, R. I.

Subscription price $3.00 a year; ordinary single numbers each $1.00, payable to the Division of Physical Sciences, National Research Council 2101 Constitution Avenue, Washington, D. C. No reductions are made to Libraries or to Booksellers.

Entered as second-class matter July 29, 1943, at the post office at Lancaster, Pennsylvania, under the Act of August 24, 1912.

and

B.)
ute

.C.

ton,

gton

New

nes,

nsin
bor,

ana,

the

ool),

City

ey
nsyl-
Ord-

gton

uncil,

mpu-
rown

vision
gton,

vania,

Optimum-Interval Punched-Card Tables

This paper describes a new principle in table-making that leads to greater efficiency in punched-card applications. Tables of the ordinary kind, proceeding by equal intervals of the argument, are often much more extensive than is necessary. In the new tables the number of entries is reduced to the minimum by allowing the interval of the argument to vary and by modifying the tabulated quantities so that interpolation is still performed in the usual way.

The use of tables in computations that are made by means of punched cards usually entails the following operations:[1] the detail cards are sorted into the table cards according to the left-hand part of the argument, the function and first difference are gang-punched from the table cards onto the detail cards, and the interpolation of the function to the value corresponding to the complete argument is accomplished by one operation of the form $F \pm nd$, with the multiplying punch. This assumes that the table is sufficiently extensive so that second differences are negligible.

In some cases a less extensive table, used with an interpolation formula that includes second differences, may be more efficient. The necessity of deriving coefficients of the second difference may be avoided if the differences in the table are replaced by the coefficients of the powers of n in the interpolation formula. Thus Bessel's formula becomes

$$\begin{aligned} F &= F_0 + n\Delta_{\frac{1}{2}}' + \frac{n(n-1)}{4}(\Delta_0'' + \Delta_1'') \\ &= F_0 + n(D_1 + nD_2), \end{aligned}$$

where $D_1 = \Delta_{\frac{1}{2}}' - \frac{1}{4}(\Delta_0'' + \Delta_1'')$, $D_2 = \frac{1}{4}(\Delta_0'' + \Delta_1'')$ are the quantities to be tabulated instead of the customary first and second differences. This principle can easily be extended to higher orders of differences.

The method just described is most efficient in application when the second differences are large and when the third differences may be neglected. When the contribution of the second differences becomes small it becomes relatively wasteful to take account of it in this way. Sometimes the difficulty may be overcome by some approximate method of sorting and gang-punching, but this is time-consuming and subject to error. The ordinary alternative is to extend the argument one more decimal place and to increase the number of cards in the table tenfold, a procedure which is even more wasteful and inefficient.

Our purpose here is to describe a much more powerful expedient, the *optimum-interval method*. The interval of the argument is allowed to vary throughout the table, the criterion being that it shall be just small enough at every point so that linear interpolation will be legitimate, with the result that the number of cards in the table is greatly reduced. This advantage is obtained without complicating the use of the table in any way, and in spite of the unusual tabular intervals.

In using any ordinary table the interpolating factor, n, is obtained by subtracting the tabular argument from the given argument, and by dividing

this difference by the tabular interval of the argument. When the tabular interval is an integral power of 10, then n is simply those figures of the complete argument beyond the position corresponding to the last digit of the tabulated argument. When the interval of the argument is not an integral power of 10, let us suppose all the differences of the table to be divided by the interval of the argument. Form the product of this modified difference by those figures of the complete argument beyond and including the position corresponding to the last digit of the tabulated argument. If this product is added to the function, the result differs from the correct interpolate by an amount equal to the product of the modified difference times the last figure of the tabulated argument. By modifying each function in the table to the extent of deducting this product, the resulting table will admit of direct interpolation in the usual manner in spite of the odd interval; it is only necessary to retain one more figure in the multiplier.

The value of the interval, ω, is determined by the equation $\Delta'' = \omega^2 d^2f/dx^2$, where $\Delta'' = 4$, if the neglected second difference effect is not to exceed one-half unit. The sequence of intervals must be chosen so that every argument the last figure of which is zero is retained in the table. Otherwise there will be cases in which the full amount by which the function has been modified will not be restored. Of all the possible combinations of intervals, the following four (in addition to the unit interval) are all that it is necessary to employ in order to obtain the advantages of the method: 2, 2, 2, 2, 2; 2, 3, 2, 3; 3, 3, 4; 5, 5.

A table of this kind differs from the ordinary table in respect of the accuracy attainable with a given number of figures. If the quantities in an ordinary table are accurate to 0.5 unit and if the second differences are negligible it is well known that a rounded-off interpolate has a maximum error of 1.0. The differences in the modified table are multiplied by factors larger than unity and the error of the interpolate depends on the interval of tabulation. The maximum rounding-off error is $\frac{1}{2}(1 + p/q)$ units, where p is the largest interval of the argument and q is the unit of the argument. In many applications an error of this size is not important but if it can not be tolerated the remedy is to carry the values of the modified functions and differences to one more significant figure. In punched-card applications the additional figure usually does not complicate the use of the table or reduce the efficiency.

To illustrate the advantages of optimum-interval tables we have prepared a six-place table of reciprocals, portions of which are shown below. The table is actually carried to seven places in order to insure six-place accuracy in the interpolates, and the tabular intervals have been chosen so as to keep the maximum error due to neglect of the second difference less than 5 units of the seventh place. Since the function is x^{-1}, $\Delta'' = 2\omega^2 x^{-3} = 4$ or $x^3 = \omega^2/2$. Taking $\omega = 2, 3, 4, 5, 10, 20, 30, 40$, in succession, the corresponding values of the argument at which these intervals may be employed are, with sufficient approximation, 1.260, 1.660, 2.000, 2.320, 3.700, 6.000, 7.700, 9.300, respectively. Let us consider a given argument lying between 1264 and 1266. In the unmodified table it is necessary first to subtract 1264 from the given argument and to divide by 2 in order to obtain the interpolating factor n. In the modified table the differences are already divided by the interval. If the end figures of the argument beyond 126 are used as

an interpolating factor the result will be in error by $4D$, and this amount has been subtracted algebraically from the true reciprocal of 1264 in order to obtain the function of the modified table. Thus the reciprocal of 12655 is $7936393 - 5.5 \times 6249 = 790202$. It is advisable in practice to add 5 to every entry of the table in order to provide for automatic rounding-off of the interpolates to six figures and this has been done here.

The total number of cards in the table of reciprocals is 1368, as compared with 9000 in the usual table, thus affording a saving of $2\frac{1}{2}$ hours of sorting and gang-punching each time the table is used. The loss due to an extra figure in the multiplier is one hour for each 10,000 detail cards. In the range from 1000 to 3700 and 370 to 600, the multiplier contains all the figures of the argument beyond the third figure, and from 600 to 999 it contains all beyond the second. In the range from 370 to 600, the modified table does not differ from an ordinary table.

Another example of the improved efficiency possible with the use of optimum-interval tables is a table of seven-place sines and cosines. The usual table, with a tabular interval of 0°.01, contains 9000 cards. The modified table contains 2700 cards, with intervals 0°.03, 0°.03, 0°.04. The maximum contribution of the neglected second difference is 0.6 unit of the seventh decimal. Each time this table is used, more than two hours of sorting and gang-punching time is saved.

The application of this principle will be found of value in all cases of punched-card tables that differ considerably from linearity over the whole range of the argument.

Arg.	Unmodified Table		Modified Table	
x	$1/x$	$-\Delta'$	f	$-D$
1250	8000005	6395	8000005	6395
1251	7993610	6385	7999995	6385
1252	7987225	6374	7999973	6374
1253	7980851	6364	7999943	6364
1254	7974487	6355	7999907	6355
1255	7968132	6344	7999852	6344
1256	7961788	6334	7999792	6334
1257	7955454	6323	7999715	6323
1258	7949131	6314	7999643	6314
1259	7942817	6304	7999553	6304
1260	7936513	12578	7936513	6289
1262	7923935	12538	7936473	6269
1264	7911397	12498	7936393	6249
1266	7898899	12459	7936279	6230
1268	7886440	12419	7936120	6210
1270	7874021	12381	7874021	6191
....	....	....	....	....
....	....	....	....	....
1990	5025131	5046	5025131	2523
1992	5020085	7549	5025117	2516
1995	5012536	5020	5025086	2510
1997	5007516	7511	5025044	2504
2000	5000005	7489	5000005	2496
2003	4992516	7466	4999983	2489
2006	4985050	9921	4999930	2480
2010	4975129	7414	4975129	2471
....	....	....	....	....
....	....	....	....	....
3690	2710032	3667	2710032	733
3695	2706365	3657	2710020	731
370	2702708	7285	2702708	7285

Arg.	Unmodified Table		Modified Table	
x	$1/x$	$-\Delta'$	f	$-D$
371	2695423	7246	2695423	7246
...	....	....	....	....
...	....	....	....	....
597	1675047	2801	1675047	2801
598	1672246	2792	1672246	2792
599	1669454	2782	1669454	2782
600	1666672	5537	1666672	2769
602	1661135	5501	1666637	2751
604	1655634	5464	1666562	2732
606	1650170	5428	1666454	2714
608	1644742	5393	1666318	2697
610	1639349	5357	1639349	2679

PAUL HERGET and G. M. CLEMENCE

U. S. Naval Observatory,
Washington, D. C.

[1] Compare W. J. ECKERT, *Punched Card Methods in Scientific Computation*, New York, The Thomas J. Watson Astronomical Computing Bureau, Columbia University, 1940.

RECENT MATHEMATICAL TABLES

138[A].—MATHEMATICAL TABLES PROJECT, New York, A. N. LOWAN, technical director, *Table of Reciprocals of the Integers from* 100,000 *through* 200,009. New York, Columbia University Press, 1943, viii, 201 p. 19.6 × 26.5 cm. Reproduced by a photo offset process. $4.00.

This work is designed to provide a 7-place table of reciprocals between 100,000 and 200,000, which will "expand by tenfold the scope of the existing tables in this interval."

The table is presented in what James Henderson has called the "modern arrangement" where the first five figures of the argument proceed in natural order down the page and the final figures from 0 to 9 are given along the top. Only the significant figures in the reciprocals are recorded. Since the entries are close together the differences vary from 100 at the beginning of the table to 25 at the end. Tables of proportional parts are given at the bottom of each page.

According to the preface we learn that: "Preparation of manuscript tables was begun in December, 1934, by Dr. C. C. Kiess of the National Bureau of Standards. The work was completed under his direction in May, 1939, with the co-operation, at various times, of Messrs. B. F. Scribner, H. R. Mullin, W. G. Esmond, and J. Waldron."

It may be well to add a short account of the development of tables of reciprocals, which have been in the making for more than a century. The first adequate table appears to have been that of PETER BARLOW, published in the first edition of his *Tables* in 1814 [*MTAC*, p. 17], where the reciprocals are given for the first 10,000 integers. A. De Morgan, in the edition of 1840, says: "I cannot ascertain that any tables of square roots, cube roots or reciprocals comparable in extent to those of Mr. Barlow were ever printed before his." In the fourth edition, issued in 1941 by L. J. Comrie, the reciprocals were extended from 10,000 to 12,500. Comrie found "60 errors of a unit in the last decimal, but none greater" in the 1840 edition.

The next significant contribution was R. PICARTE'S, *La division réduite à une Addition*, Paris, 1861, which provided values of the reciprocals from 1000 to 10,000 to 10 significant figures, together with the first nine multiples of them.

W. H. OAKES published his 7-place *Table of the Reciprocals of Numbers from 1 to 100,000*, in London in 1865. The volume under review is a continuation of this work, which was similarly printed in the "modern arrangement," with differences at the side of the page. The more recent table of M. B. COTSWORTH, erroneously entitled "Reciprocals for All Numbers from 1 to 10,000,000," [1] is also a 7-figure table over the same range as that of Oakes.

J. C. Houzeau in Acad. r. des Sciences . . . de Belgique, *Bull.*, v. 40, 1875, p. 107, published a 20-place table of reciprocals of the first 100 integers and their first 9 multiples to 12 places. Apparently ignorant of this work, Karl Pearson, in v. 2 of his *Tables for Statisticians and Biometricians*, London, 1931, p. 257 gave 20-decimal values of the reciprocals to 100. Ten-place reciprocals of the first 1,000 numbers are also given by E. Gélin in his *Recueil de tables numériques*, Huy, 1895.

Mention has already been made in this journal (p. 164) of the 17-figure tables for the reciprocals of the first 1,000 integers which are in the possession of the reviewer. He has recently acquired a 9-figure table of the reciprocals of numbers from 1 to 10,000, the work of Charles Mills of Northwestern University.

H. T. D.

[1] A misstatement in the book under review. A more correct title of the work in question is as follows: *Cotsworth's Reciprocals for All Numbers From* 1 *to* 10,000,000 *with Direct Index Reference to Seven Significant Figures, complete for Numbers under* 100,000 *beyond which the extension to* 10,000,000 *is rendered easy by the Additive Differences printed at the foot of each Table*, York, England, M. B. Cotsworth, 1902.—Editors.

139[C].—H. S. Uhler, "Natural logarithms of small prime numbers," Nat. Acad. Sci., *Proc.*, v. 29, 1943, p. 319–325. 17.1 × 25.6 cm.

The results here given are supplementary to those in Mr. Uhler's publications already reviewed in *MTAC*, p. 55, 56, 20 (RMT **95**, **96**, **86**). It was thus already noted that J. C. Adams (1887) calculated each of the quantities ln2, ln3, ln5, and ln7, to 272D; and that Mr. Uhler extended each of these to 328D and found ln17 to 331D, and ln71 to 213D. In the present paper the following new results are presented: ln11 (329D), ln13 (290D), ln19 (299D), ln23 (295D), ln29 (300D), ln31 (292D), ln37 (291D), ln101 (329D), and ln9901 (303D). Furthermore, each of the following is given to 155D and is supplementary to T. 4 of the monograph on 137-place values of $\ln(1 \pm n \cdot 10^{-p})$, RMT **86**: ln41, ln43, ln59, ln61, ln67, ln73, ln79, ln83, and ln89. Thus for every N, prime and < 100, $\ln N$ is given to at least 155D. The descriptions of checks applied tend to inspire confidence in the accuracy of the results. The author wished us to add, however, that in ln43, line 3, 12917, *should read* 12971; a slip due to failure of the printer to make this correction, indicated in proof; there is no error in the reprints.

140[C, L].—Walter Meissner, *Tafel der* ln Γ*-Function mit komplexem Argumentbereich*, Dresden Diss., *Deutsche Mathematik*, v. 4, 1939, p. 537–555. 20.8 × 28.9 cm.

For the 192 complex arguments

$$z = (2m + i\,2n\sqrt{3})/24, \quad \text{and} \quad z = [(2m+1) + i(2n+1)\sqrt{3}]/24,$$

with $m = 6, 7, \cdots, 16, 17$; $n = 0, 1, \cdots, 6, 7$ (triangular net), and for the gamma function

$$\Gamma(z) = e^{u+iv} = 10^{\lambda} \cdot e^{i\phi},$$

u and v are given to 7D (except for 24 entries to 15D); $\lambda = u \log e$ is given to 7D, and $\phi = v180/\pi$ is expressed in degrees, minutes, and whole seconds. (See F. Emde, *Z. angew. Math. Mech.*, v. 20, 1940, p. 295). The tables were intended as a supplement to the little tables of $\Gamma(z)$ and $\ln \Gamma(z)$ (p. 111, 109), z real, in the chapter on gamma functions in P. E. Böhmer, *Differenzengleichungen und bestimmte Integrale*, Leipzig, Köhler, 1939, where the complex case is discussed.

R. C. A.

141[D].—J. C. P. Miller, *Tables for Converting Rectangular to Polar Coordinates*. London, Scientific Computing Service, 23 Bedford Square, London W. C. 1, 1939, 16 p. 15.3 × 25 cm. Two shillings. Authorized American reprint, Dover Publications, 31 East 27th St., New York 16 [1943]. Seventy-five cents.

The main object of these excellent tables is to facilitate the conversion of rectangular coordinates (x, y) to polar coordinates (r, θ) by means of the relations $r = (x^2 + y^2)^{\frac{1}{2}}$, $\theta = \tan^{-1}(y/x)$.

Denote by l the larger and by s the smaller of $|x|$ and $|y|$, and evaluate $k = s/l$; this is the argument of the tables and evidently lies in the range 0 to 1. Then we have $r = (l^2 + s^2)^{\frac{1}{2}} = l(1 + k^2)^{\frac{1}{2}}$; and $\theta_0 = \tan^{-1} k = \tan^{-1}(s/l)$, $\phi_0 = \cot^{-1} k = \cot^{-1}(s/l) = 90° - \theta°$. Then $\theta = \tan^{-1}(y/x)$ may be determined in terms of θ_0 or ϕ_0 if the result is required in degrees or in terms of θ_0 alone if radians are used, with the help of an "octant" scheme, in which the signs and relative size of x and y constitute the argument.

The small interval of tabulation, namely 0.001, leads to the advantage that the difference between two successive values of any function never exceeds 10; interpolation may therefore be performed mentally. For cases where r is required to three significant figures only, and θ to $0°.1$ only, interpolation is unnecessary.

The text discusses "Method with a calculating machine," "Slide rule method" and "Applications of the tables." Under the last heading it is noted that the tables may be used for transformation of harmonic constants a and b, obtained by harmonic analysis, to amplitude c and phase angle ϵ, in accordance with the relation

$$c \sin(nt + \epsilon) = a \sin nt + b \cos nt.$$

The use of the tables for the evaluation of impedance and phase lag or lead from resistance and reactance will immediately suggest itself to electrical engineers. The evaluation of the magnitude and direction of a vector from rectangular components and the conversion of complex numbers from the form $x + iy$ to the form $r = e^{i\theta}$ are further applications.

142[D, E].—MATHEMATICAL TABLES PROJECT, New York, A. N. LOWAN, technical director, *Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments.* New York, Columbia University Press, 1943, xxxviii, 410 p. 19.6 × 26.5 cm. Reproduced by a photo offset process. $5.00.

The main table of this interesting volume is a 400-page table of $\tan x$, $\cot x$, $\tanh x$, $\coth x$, for $x = [0.0000(0.0001)2.000]$ radians. The number of decimal places varies from 5 to 13 for the different functions and the different argument ranges. The greater part of the table gives values to 8 significant figures. Near poles more significant figures are given as noted below. There are auxiliary tables as follows: Table II gives these functions for $x = [0.0(0.1)10.0$ radians; 10D], conversion tables: degrees, minutes, and seconds to radians and vice versa,[1] integral multiples of $\pi/2$ to 15D. Interpolation coefficients $p(1 - p)/2$ and $p(1 - p^2)/6$ for $p = [0.000(0.001)1.000; 6D]$.

Except near $x = 0$ and $x = \pi/2$, the main table was constructed, one entry at a time, by actual division of the corresponding values of $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, taken from this Project's table of circular and hyperbolic sines and cosines.[2] The values thus obtained were then verified by the reciprocal relations

$$\tan x \cot x = \tanh x \coth x = 1.$$

The resulting table was then differenced, to reveal errors in copying into the manuscript. This meticulous procedure, characteristic of this Project's fine large tables, was made possible by the large man- and machine-power available. Although the computer, to whom a good computing machine is available, can calculate isolated entries in the same way, or better still for the hyperbolic functions, by use of the formulas

$$\tanh x = (e^{2x} - 1)/(e^{2x} + 1), \qquad \coth x = (e^{2x} + 1)/(e^{2x} - 1),$$

where the value of e^{2x} can be taken from this Project's excellent table of the exponential function,[3] nevertheless the present table is more convenient (especially for the circular functions) when some interpolation is required.

The arrangement of the function values is a little confusing to the reader who merely thumbs the pages. The last three digits of each entry are separated from the earlier digits by a space. This makes for a variable number of digits between this space and the decimal, an irritating and perhaps hazardous circumstance. For $x = [0.0000(0.0001)0.1000]$, all four functions are given to 8 decimals. Since $\cot x$ and $\coth x$ are quite large here, these functions are given with great accuracy, and since they are decreasing rapidly, interpolation is difficult. For the sub-range $x = [0.0300(0.0001)0.1000]$ a column of second differences is provided ranging from 74075 to 2000 units of the 8th decimal place. For still smaller values of x direct interpolation is impractical. The rest of the main table gives the four functions to 8 significant figures except for page 316 ($x = [1.5700(0.0001)1.5750]$) where $\cot x$ is given to 13 decimal places. On p. viii of the foreword, by H. T. D., one reads "The figures seen on the printed page may appear dull reading to the uninitiated. . . ." For once, this is not true of p. 316. Here one sees the function $\tan x$ passing through its agonizing crisis at $x = \pi/2$, its onetime companion $\tanh x$ moving slowly ahead with apparent unconcern. Striking as these values of $\tan x$ are, they are of little practical value. To use any one of them, the value of x must be known with extreme certainty. Interpolation is not possible. Perhaps it would have been more practical to tabulate, near $\pi/2$, the function

$$\tan x - (\pi/2 - x)^{-1}$$

which is interpolable. Similarly near $x = 0$ the functions

$$\cot x - 1/x, \qquad \coth x - 1/x,$$

might have been tabulated. The policy of basing the table on 8 significant figures rather than a fixed number of decimals, is presumably to make uniform the relative error, rather than the actual error in the tabular entries. This expedient fails to produce the desired result in the case of $\tanh x$ and $\coth x$. Near the end of the table the $\tanh x$ values are, on the average, nearly ten times as accurate as those of $\coth x$, on the basis either of relative or actual error (since both functions are nearly unity). For the last half of the table, the relative error in $\tanh x$ averages about sixteen percent that of $\coth x$.

The introduction contains (p. xxiii–xxxviii) a bibliography of tables and charts of circular and hyperbolic tangents and cotangents and their logarithms, the inverse hyperbolic functions, the gudermannian and related functions, together with tables of roots of equations involving $\tan x$. No short bibliography could cover the very large number of titles of this description. In the present case only 73 important items are listed. These do not include any tables produced by New York Tables Project. Of these 73 tables only 17 refer to circular or hyperbolic tangents or cotangents given to more than 6 decimal places. Of these 17, only one table is for radian measure. This is the unreliable table of HAYASHI[4] in which $\tan x$ and $\tanh x$ are tabulated up to $x = 50$, the main table being for $x = [0.0010(0.0001)0.100(0.001)3.00(0.01)9.99]$ Thus the table under review fills a real gap in the bibliography of these elementary functions. It will, no doubt, be useful chiefly in problems in which a large number of values of one of the four functions are needed quickly. The fact that linear interpolation gives such good results over most of the table makes it quite useful to the engineer or physicist. It will help to wean away the typical astronomer and civil engineer from his sexagesimal table of log tan x.

D. H. L.

[1] In the reviewer's opinion such conversion tables should be used only for pencil and paper work. With a machine available they are too slow and dangerous to bother with.
[2] *Tables of Circular and Hyperbolic Sines and Cosines*, New York, 1939.
[3] *Tables of the Exponential Function* e^x, New York, 1939.
[4] K. HAYASHI, *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen und deren Produkte sowie der Gammafunktion*, Berlin, Springer, 1926.

NOTE.—R. C. A. requests that his statement concerning works of IVES and BENSON in reviews of *MTAC*, 1943, p. 8–10, be considered rather than the one quoted on p. xxx, l. 16–20, from *Scripta Mathematica*, v. 2, 1933, p. 91.

143[D, E, L].—HERBERT BRISTOL DWIGHT (1885–   ), *Mathematical Tables of Elementary and Some Higher Mathematical Functions including Trigonometric Functions of Decimals of Degrees and Logarithms*, New York and London, McGraw-Hill Book Co., 1941. Third impression (with additions), 1944, viii, 231 + 8 p. 15.2 × 22.8 cm. $2.50. Reproduced by a photo offset process.

A detailed statement of the contents of this volume will best exhibit its value. First differences for interpolation are given in connection with practically all of the tables. After the tables are many references to sources where more extended tables may be consulted. Tables I–V: sine, cosine, tangent, cotangent and their logarithms, and also secant and cosecant for every hundredth of a degree, to 5D, p. 1–101. T. VI–VIII: sine, cosine, for each $0^r.001$, 0 to 2, and tangent, 0 to 1.570, p. 102–113; T. IX–XI: $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $x = [0^r.000(0^r.001)1^r.000; 4D]$ and $\tan^{-1} x$ for more extended values, p. 114–121; T. XII: $\ln x$, $x = [1.000(0.001)3.00(0.01)10; 4D]$, p. 122–127; T. XIII–XIV, e^x, e^{-x}, $x = [0.000(0.001)5; 4\text{–}6S]$, p. 128–147; T. XV–XVII: $\sinh x$, $\cosh x$, $\tanh x$, $x = [0.000(0.001)3; 4\text{–}6S]$, p. 148–165; T. XVIII–XIX: $\sinh^{-1} x$, $\cosh^{-1} x$, $x = [0.000(0.001)2.00(0.01)10; 4\text{–}5S]$, p. 166–175; T. XX: $\tanh^{-1} x$, $x = [0.000(0.001)1; 4\text{–}5S]$, p. 176–178; T. XXI: binomial coefficients ${}_1C_r$ to ${}_{25}C_r$, p. 179; T. XXII: $(a^2 + b^2)^{\frac{1}{2}}/a$, $b/a = 0.000(0.001)1$, with a reference to Miller's *Tables* (see RMT **141**), p. 180–181; T. XXIII: factorials $n!$, $n = 1$ to 50, 1–6S, p. 182; T. XXIV: $\log n!$, $n = 1$ to 250, 5–7D, p. 182–183; T. XXV: Gregory-Newton interpolation coefficients, ${}_pC_r$, $r = 2(1)6$, $p = [0.01(0.01)1.00; 2\text{–}7D]$, p. 184–185; Lagrangean interpolation coefficients for interpolating without columns of differences, p. 186–187; T. XXVI–XXVII: surface zonal harmonics $P_n(x)$, $P_n(\cos\theta)$, $n = 1(1)10$, $x = [0.00(0.01)1; 1\text{–}6D]$, $\theta = 0°(1°)90°$; p. 188–195; T. XXVIII: first derivatives of the surface zonal harmonics $P_n(\cos\theta)$, $\theta = 0°(1°)90°$ from Farr's article, and therefore very erroneous (see MTE **33**), p. 196–197; T. XXIX: complete elliptic integral of the first kind, p. 199–203; T. XXX: complete elliptic integral of the second kind, p. 204–205; T. XXXI: Bernoulli's numbers (35), p. 206; T. XXXII: Euler's numbers (35), p. 207; T. XXXIII: $\Gamma(n)$, $n = [1.00(0.01)2; 5D]$, p. 208–209; T. XXXIV: erf x, $x = [0.000(0.001)1.50(0.01)3; 5D]$, p. 212–213; T. XXXV: ber x, bei x, ber′ x, bei′ x, p. 214–217; T. XXXVI: ber x + ibei x, ber′ x + ibei′ x, p. 218–221; T. XXXVII–XXXIX: Riemann zeta function $\zeta(s)$, $s = [-24.0(0\text{–}1)24; 10\text{–}11S]$, also tables of $(s-1)\zeta(s)$ with δ^2, δ^4 and $\zeta'(s)/\zeta(s)$, p. 222–227; T. XXXX: $\log x$, $x = [1.00(0.01)9.99; 4D]$, p. 228–229; Index, p. 231. The zeta function tables were furnished by H.T.D.

Such are the contents of the first edition. In the current edition Preface and Contents are printed from type rather than "offset"; literature notes have been modified on pages 195, 197 and 213; on p. 124 ln 2·269, for ·8193 has been substituted .8194; argument headings on p. 208–209 have been corrected; and 8 pages have been added. On the new p. 195A-B are the first derivatives of surface zonal harmonics $P'_n(x)$, $n = 2(1)8$, $x = [0.00(0.01)1.00; 2\text{–}6S]$; the erroneous p. 196–197 persist. Six new pages, 213A-F, are devoted to a table of the normal probability integral.

144[F].—HANSRAJ GUPTA, "On the class-numbers of binary quadratic forms," Tucuman, Argentina, Universidad, *Revista*, s. A, *Matem. y Fisica Teorica*, v. 3, 1942, p. 283–299. 17.8 × 27.1 cm.

An integral binary quadratic form is a homogeneous expression

$$f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 = [a, b, c] \tag{1}$$

with integral coefficients a, b, c. If the greatest common divisor (a, b, c) of a, b, and c is equal to unity, then f is said to be properly primitive. A binary quadratic form $F(y_1, y_2)$ obtained from $f(x_1, x_2)$ by a substitution $x_i = \alpha_{i1}y_1 + \alpha_{i2}y_2$ $(i = 1, 2)$ with integral coefficients and of determinant $\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12} = 1$, is said to be properly equivalent to f. Both

f and F have the same discriminant $b^2 - 4ac = -d$. All properly equivalent forms are said to form a class.

The present paper contains a table of the number $h'(d)$ of the classes of properly primitive binary quadratic forms of negative discriminant $-d$, $1 \leqslant d \leqslant 12500$, together with the description of the procedure employed in the computation. It is the most extensive class-number table known to the reviewer. However, it contains no such related information on genera as do, for example, the extensive tables of Gauss.[1] The total number $h(d)$ of classes of discriminant $-d$ is obtained from the well-known formula

$$h(d) = \sum_{k^2|d} h'(d/k^2).$$

There exist closed expressions[2] of theoretical importance for $h'(d)$. None of these, however, are really suitable for computation. The method of construction of the present table is based on the following considerations. As is well known, every properly primitive quadratic form of negative discriminant $-d$ is equivalent to one and only one *reduced* form f for which[3]

$$(2) \qquad -a < b \leqslant a, \qquad 0 < a \leqslant c, \qquad b > 0 \text{ if } a = c, \qquad (a, b, c) = 1.$$

Thus the number $h'(d)$ is the same as the number of properly primitive reduced forms of discriminant $-d$. Should one know the number[4] $D(d, B)$ of reduced forms (2) with $|b| = B$, one may use the relation

$$h'(d) = \sum D(d, B), \qquad 0 \leqslant B \leqslant [d^{1/2}/3]$$

to obtain $h'(d)$.

To compute $D(d, B)$ it suffices to determine the number $N(i, B)$ of decompositions $i = (d + b^2)/4 = a \cdot c$ into two positive factors a and c such that

$$(3) \qquad |b| < a < \sqrt{i}, \qquad (C(a, c), b) = 1,$$

where $C(a, c) = (a, c)$.

For, $D(d, 0) = N(d/4, 0)$. Next let $B \neq 0$.

In case i is not a perfect square, then

$$(4) \qquad D(d, B) = 2N(i, B) + 1, \qquad \text{if } B|i \text{ and } C(B, c) = (B, i/B) = 1,$$

and

$$(5) \qquad D(d, B) = 2N(i, B) \text{ otherwise.}$$

In case i is a perfect square, then formulae (4) and (5) hold if $(B, i) > 1$. If, however, $(B, i) = 1$, the values in (4) and (5) must be increased by $+1$. Thus, in this case:

(6) $D(d, 1) = 2N(i, B) + 2$; $D(d, B) = 2N + 1$ if $B > 1$ and either $(B, i) = 1$ or $B|i$ and $(B, i/B) = 1$; $D(d, B) = 2N$ for other values of B.

These formulae follow at once from (2). The term $+1$ in (4), accounts for the reduced form with $a = B = b$ which is properly primitive only if $C(a, c) = 1$. The unit to be added when i is a perfect square accounts for the form with $a = c$, $b > 0$. This form is properly primitive if $(B, a) = (B, a^2) = (B, i) = 1$. Actual computation may be arranged systematically as follows. First comes the "preliminary table."

i	a, B	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	$C(a, c)$		1	2		4			1	4						2	
224	$N(i, B)$[5]	2	5	1	4	1	3	1	2	0	1	0	1	0	1	0	
	$D(4i-B^2, B)$	2	11	2	8	2	6	2	5	0	2	0	2	0	2	0	
	C		1		3		5				1						15
225	N [5]	2	3	3	2	2	1	1	1	1	0	0	0	0	0	0	0
	D	2	8	7	4	5	2	2	3	3	1	0	1	0	1	1	0

Here, in view of (3), $N(i, B)$ is obtained by counting the number of $C(a, c)$ with $B < a < i^{1/2}$, which are prime to B. The value of D is obtained through the formulae (4)–(6).

Next, the D entries are copied out opposite the proper values of d in another table and are totalled by rows to give $h'(d)$.

Thus, the method of tabulation used by the author is essentially the usual method of construction of the table of reduced forms in which the scheme of computation is so arranged that only the presence of a reduced form $[a, b, c]$ is recorded and not the form itself. One observes with interest that the desirable scheme in this case differs from the excellent arrangement by Wright[6] in tabulating reduced forms themselves. As in Wright the method calls for only a simple succession of elementary operations.

St. Louis University ARNOLD E. ROSS

[1] C. F. GAUSS, Werke, v. 2, p. 450–476. For reviews of this and other similar tables consult D. H. LEHMER, *Guide to Tables in the Theory of Numbers*, Nat. Res. Council, *Bull.* no. 105, 1941.

[2] Compare L. E. DICKSON, *History of the Theory of Numbers*, v. 3, Washington, D. C., 1923 (New York reprint, 1934), chapter VI.

[3] Compare L. E. DICKSON, *Modern Elementary Theory of Numbers*, Chicago, 1939.

[4] The author uses $D((d+B^2)/4, B)$.

[5] The author omits this row and records the values of D directly.

[6] H. N. WRIGHT, "On a tabulation of reduced binary quadratic forms of a negative determinant," California, University, *Publs. in Math.*, v. 1, no. 5, 1914, p. 98–114+2 folding plates.

145[F].—CHAO KO and S. C. WANG, "Table of primitive positive quaternary quadratic forms with determinants $\leqslant 25$," Academia Sinica, *Science Record*, v. 1, nos. 1–2, Aug., 1942, p. 54–58, Chungking, China. [In the 260 pages of this publication 97 pages are devoted to 25 papers announcing results of mathematical research.] 18 × 25.4 cm.

In the study of diophantine equations (1) $f(x) = m$, where $f(x) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ is a quadratic form in n variables $x_1, \cdots, x_n$ with integral coefficients a_{ij}, it was noticed at an early date that integral solutions of (1) may be obtained from the solutions of a related equation (2) $F(y) = m$, where $F(y)$ is a quadratic form *equivalent* to $f(x)$, i.e., a form obtained from f by a transformation τ: $x_i = \sum_{j=1}^{n} \tau_{ij}y_j$ with integral coefficients τ_{ij} and of determinant $|\tau_{ij}|$ equal to unity. Thus, seeking solutions of (1) and (2) independently would involve needless duplication of effort which would be avoided if one would select for study only one of the unlimited number of equations (1) whose left-hand members f are equivalent. A process of selection of a limited number of representative forms, desirably a single one, for each set (*class*) of equivalent forms is called *reduction*, and the selected representatives are referred to as *reduced* forms. Since the number of classes of quadratic forms of a fixed determinant $|a_{ij}|$ is finite, a limited number of reduced forms suffices to represent all forms of a given determinant.

In the paper under review, the authors have constructed a table of *primitive* (*g c d* of $2a_{ij}$ and a_{ii} is unity) reduced positive quaternary quadratic forms of determinant $D \leqslant 25$, using a method of reduction which gave exactly one representative for each class of equivalent forms. Such a table should be considered an improvement of CHARVE'S[1] original table, since his method of reduction led to more than one reduced form for some classes. The authors were apparently unaware, however, of the existence of a similar table by S. B. TOWNES[2], $D \leqslant 25$. Townes's method of reduction employs ideas of Eisenstein and also leads to a unique representative. Reduced forms in the two tables differ only in the choice of the cross-product coefficients.

There are a number of errata in the article under review; on p. 56f, $D = 20$, *for* (1,1,2,20), *read* (1,1,2,10); $D = 21$, *for* $(1\,2\,3\,3\,0_4\,-1\,0)$, *read* $(1\,2\,3\,4\,0_4\,-1\,0)$; and $D = 24$, *for* $(2\,2\,8\,3\,-1\,0_3\,-1\,0)$, *read* $(2\,2\,3\,3\,-1\,0_3\,-1\,0)$. On p. 58, the fourth determinant is 9 and not 8. In addition to these errata, noted by the authors in an errata list at the end of the issue of the *Record*, we note that the entries $(2\,2\,3\,4\,-1_2\,0_3\,-1)$ for $D = 16$ and

$(2\,2\,3\,3\,0\,-1_2\,0\,-1_2)$ and $(2\,2\,3\,5\,-1_2\,0_3\,-1)$ for $D = 24$ are incorrect, being actually of determinants 28, 11 and 36 respectively. A representative for one of the twenty-four classes of determinant 24 is missing, and the entry $(2\,3\,3\,3\,-1\,0_2\,-1\,0_2)$ of $D = 25$, should read $(2\,2\,3\,3\,-1\,0_2\,-1\,0_2)$.

To restore completeness of the table one may use the following reduced forms from Townes in place of the entries referred to above: the form $(2\,2\,3\,3\,1_2\,0\,1_3)$ for $D = 16$, and the forms $(2\,2\,3\,4\,1_2\,0\,1_3)$, $(2\,3\,3\,3\,-1_2\,1\,0\,1_2)$, and $(1\,2\,4\,4\,0_5\,2)$ for $D = 24$.

The count of the number of classes yields 14 for $D = 16$ and 24 for $D = 24$. We have, in the above, used the well-known notation $(a\,b\,c\,d\,r\,s\,t\,u\,v\,w) = ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2 + 2rx_2x_3 + 2sx_1x_3 + 2tx_1x_2 + 2ux_1x_4 + 2vx_2x_4 + 2wx_3x_4$. A run of k zeros is indicated by 0_k for brevity. One may note that the entry $(1\,1\,4\,4\,0_5\,2)$ for $D = 16$ in Townes's table should read $(1\,1\,4\,5\,0_5\,2)$.

St. Louis University ARNOLD E. ROSS

[1] L. CHARVE, "Table des formes quadratiques quaternaires positives réduites dont le déterminant est égal ou inférieur à 20," Acad. d. Sci., Paris, *Comptes Rendus*, v. 96, 1883, p. 773–775.

[2] S. B. TOWNES, "Table of reduced positive quaternary quadratic forms," *Annals Math.*, s. 2, v. 41, 1940, p. 57–58.

EDITORIAL NOTE: Mr. Ross was formerly called ARNOLD CHAIMOVICH, as we learn from his mimeographed English translation from the Russian of P.S. NAZIMOV's Prize Essay (Moscow, 1884): *Applications of the Theory of Elliptic Functions to the Theory of Numbers*, Evanston, Ill., 1928.

146[F].—D. H. LEHMER, "Ramanujan's function $\tau(n)$," *Duke Math. J.*, v. 10, 1943, p. 483–492.

The numerical function $\tau(n)$ is the coefficient of x^n in the power series expansion of $x[(1-x)(1-x^2)(1-x^3)\cdots]^{24}$ This function has long been of interest in the theory of numbers, particularly in the arithmetical applications of the elliptic modular functions. If m, n are coprime,

$$\tau(mn) = \tau(m)\tau(n); \tag{1}$$

if p is prime

$$\tau(p^{\alpha+1}) = \tau(p)\tau(p^\alpha) - p^{11}\tau(p^{\alpha-1}), \quad \alpha \geq 1. \tag{2}$$

Hence $\tau(n)$ is calculable from the values of $\tau(p^\alpha)$, which in turn are calculable for $\alpha = 2, 3, \cdots$ when $\tau(p)$ is known. An explicit formula for $\tau(p)$ is lacking. Ramanujan conjectured (from scanty numerical evidence, namely, the 10 cases determined by $p < 30$) that $|\tau(p)| < 2p^{11/2}$, so that $p^{-11/2}\tau(p) = 2\cos\theta_p$, θ_p real. This unproved conjecture, whose truth was regarded by Ramanujan as "highly probable," is called by Hardy the "Ramanujan hypothesis." The author's purpose is best described in his own words: "In seeking to disprove the Ramanujan hypothesis I have examined all primes $p < 300$, that is, the first 46 primes, as well as $p = 571$, and I find that in all these cases the hypothesis is true. There is only one 'near miss' which occurs at $p = 103$."

The table gives the value of $\tau(n)$ for $n \leq 300$. For composite n the values were computed by (1), (2). For n prime a recurrence formula in which the arguments of $\tau(n)$ decrease by successive pentagonal numbers is used. The formula is obtained by an application of Euler's power series expansion of $\Pi(1 - x^n)$, $n = 1, 2, \cdots$ to the logarithmic derivative of the generating identity for $\tau(n)$, and is

$$(n-1)\tau(n) = \Sigma[n - 1 - \tfrac{1}{2}\,25(3m^2 + m)]\,\tau\,[n - \tfrac{1}{2}(3m^2 + m)],$$

the summation referring to $1 \leq |m| < a_n$, $6a_n = 1 + (1 + 24n)^{\frac{1}{2}}$. This formula has practical computational advantages over Ramanujan's recurrence formula involving triangular numbers, derived similarly from Jacobi's expansion of $\Pi(1 - x^n)^3$.

The calculations for $\tau(n)$, n prime, were checked by Ramanujan's congruence

$$\tau(n) \equiv \sigma_{11}(n) \bmod 691,$$

where $\sigma_{11}(n)$ is the sum of the 11th powers of the divisors of n. The table as a whole was checked by comparison at frequent intervals of the values modulo 691 of $\Sigma\tau(s)$, $\Sigma\sigma_{11}(s)$, $s \leq x$. Further congruences, to be discussed in another paper, also were used as checks. In addition to the table and the method of computing and checking it, the paper discusses problems connected with the orders of $\tau(n)$, $\Sigma\tau(s)$, neither of which is known.

E. T. BELL

California Institute of Technology

147[I].—ARTHUR SHEPARD LITTLE (1872–), *A Table of Interpolation Multipliers, for obtaining through the means of calculating machines, Intermediate Rates Bond Values at yield intervals of one ten-thousandth percent.* Boston, Financial Publishing Co., [1927], 29 p. 17.5 × 21.6 cm. $4.00.

The main table of this work gives the four coefficients A, B, C, D, in the following special case of Lagrange's interpolation formula:

$$f(a + 2hx) = A(x)f(a) + B(x)f(a + h) - C(x)f(a + 2h) + D(x)f(a + 3h)$$

where

$$\begin{aligned} A(x) &= -(2x - 1)(x - 1)(2x - 3)/3, \\ B(x) &= 2x(x - 1)(2x - 3), \\ C(x) &= x(2x - 1)(2x - 3), \\ D(x) &= 2x(2x - 1)(x - 1)/3, \end{aligned}$$

for $x = [0.000(0.001)0.500; 9D]$. The reason for the peculiar "$2x$" in the above is that the table is intended for use with bond tables having an argument interval of 0.05 percent. There is also on p. 29 a small table giving the three coefficients in the corresponding three-term formula:

$$f(a + 2hx) = A(x)f(a) + B(x)f(a + h) - C(x)f(a + 2h)$$

where

$$\begin{aligned} A(x) &= (2x - 1)(x - 1), \\ B(x) &= -4x(x - 1), \\ C(x) &= -x(2x - 1), \end{aligned}$$

for $x = [0.00(0.01)0.50; 4D]$. The author's 8 pages of introduction are interesting (not to say amusing) but inadequate. None of the above formulas appears in the book and not even one small example is worked out to show the business man how to use these tables. Instead there are repeated warnings against failing to notice that formulas (which are not given) contain minus signs so that the crank of the calculating machine must be turned in the negative direction at the proper time. To make this point very clear the coefficients $C(x)$ in the two tables are printed in red and the column headings bear the injunction "Turn Off," while the other columns are headed "Turn On."

The author does not indicate how he discovered these interpolation formulas. There is no reference to Lagrange or to any other source. The author believes that his method is "absolutely original and is now being made public for the first time."

The Lagrange type of interpolation formula is, I believe, destined to become more and more used by computers to whom a good calculating machine is available. The more traditional formulas employing differences are products of an age of pencil and paper calculation. Many tables are printed without differences. When the successive differences are available or worked out, the various terms of the interpolation formula vary so in size that they are a constant source of irritation if not of actual error. With the Lagrange type, the separate terms are of the same order of magnitude and are automatically accumulated in the product register. A small table of Lagrangean interpolation coefficients, such as the one under review,[1] should be in the library of every computer.

D. H. L.

[1] Perhaps one of the best elementary introductions to the subject may be found in K. PEARSON's *Tracts for Computers*, nos. II–III, Cambridge, University Press, 1920, with illustration examples and short tables of coefficients in the 4 to 11-point "Lagrangean

formulae." LAGRANGE gave the formula now associated with his name in Paris, École Polytechnique, *J.*, v. 2, 1795, p. 274–278. But this formula was discovered twice earlier, namely: (1) by EDWARD WARING, R. So. London, *Trans.*, v. 69, 1779, p. 59–67, and also in generalized forms; (2) by L. EULER, *Opuscula Analytica*, v. 1, 1783, p. 184, setting forth the general Lagrangean formula for equal intervals.

Reference may be given also to T. L. KELLEY, *The Kelley Statistical Tables*, New York, Macmillan, 1938. There are cubic (4-point, [0.000(0.001)1; 10D]), quintic (6-point, [0.00(0.01)1; 10D]), and septic (8-point, [0.0(0.1)9; 11D]) interpolation tables. See RMT **130**. And finally we may note MATHEMATICAL TABLES PROJECT, *Table of Lagrangean Coefficients*, Ordnance Department, 1941, 40 p. 22×26.5 cm. This edition of a 5-point table [0.000(0.001)2; 7D] was not available to the public until prepared for distribution by the Marchant Co., Oakland, California, in 1942. This is now published on p. 383–403 of MATHEMATICAL TABLES PROJECT, *Table of Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments*, New York, Columbia University Press, 1943; see RMT **151**. The PROJECT has in the press an elaborate volume, entitled *Tables of Lagrangian Interpolation Coefficients* (xxxii, 390 p.).

EDITORIAL NOTE.—We include here a review of this table published before 1933, for two reasons, because only the slightest attention has been paid to it in any mathematical periodical, and because there is little doubt that the author, working most of his life in a clerical capacity for the First National Bank and one of its affiliates, in St. Louis, was an independent discoverer of the utility of these so-called Lagrange interpolation coefficients. He published at least three other volumes (1915–26), listed in the Library of Congress Catalogue, and a good many articles in *Journal of Accountancy*.

148[I].—A. J. THOMPSON, *Tables of the Coefficients of Everett's Central-Difference Interpolation Formula. Tracts for Computers*, No. V. Edited by E. S. Pearson and published by the Department of Statistics, University College, London, Cambridge University Press, second ed., 1943. viii, 32 p. 16 × 23.3 cm. 5 shillings. See also MTE **41**.

The first edition of Mr. Thompson's tables was published in 1921 and was quickly sold out. The present edition differs considerably from the first (xvi, 21 p.). The size of the introduction is cut by half and its contents (somewhat paradoxically) are increased. Also there are five tables of coefficients in the new edition, the first three columns of Table I covering the whole of the single table in the original edition. The reputation of the distinguished author guarantees the accuracy of the figures published. It may be expected, therefore, that the tract will be in great demand.

A considerable portion of the original Introduction was devoted to numerical examples illustrating in varying degrees the same type of application of Everett's interpolation formula and coefficients. In the second edition the number of numerical examples is reduced to just a single one. In addition to this, the author gives to the reader the benefit of his wide experience in dealing with various difficult situations. A straight application of Everett's formula is possible when the table, in which it is desired to interpolate, contains the central differences. This, of course, is far from being always the case. A convenient method is indicated by which explicit use of the central differences can be avoided.

Other material in the new Introduction deals with the construction of mathematical tables. The first step suggested is to compute directly the values of the function to be tabulated for a basic framework of widely spaced values of the argument. These computations should yield some three or four places of decimals in excess of the accuracy of the proposed table. Once the framework is completed, the use of Tables II, III, IV and V of the new edition of Thompson's tract will permit a relatively easy process for filling of the gaps by reducing the original increment of the argument in the ratio of one to ten or more. The coefficients of Everett's formula are used to compute both the values of the function and the corresponding central differences.

Table I gives the values of the first four coefficients for values of the argument $0 \leqslant \theta \leqslant 1$ proceeding by increments of .001. It also gives the central differences of the coefficients which could be used for interpolation among the values published. Table II contains coefficients of the same orders as Table I but proceeding at intervals of θ equal to .01. Table III gives the values of six coefficients, ϵ_2 to ϵ_{12}, with complete differences, but proceeds at

increments of θ equal to .1. Table IV has the same order of coefficients and the same increment of θ as Table III but the range of θ is from -5 to $+6$. The primary purpose of this table is to facilitate interpolation in other tables near the ends of the range of their arguments, where the central differences of higher orders cannot be made available without previously extending the range of the original table. In Table V the coefficients extend to ϵ_{16} and proceed at increments of θ equal to .2.

Those who have frequent occasions to interpolate in tables in which linear or quadratic interpolation is inadequate, and have had to do so without the benefit of Mr. Thompson's tract, may think that its title is not satisfactory and that it should be changed to "Interpolation Made a Pleasure," or to "A Hard Life Made Easy."

J. NEYMAN

University of California, Berkeley

149[I, J].—H. E. SALZER, "Coefficients for numerical differentiation with central differences," *J. Math. Phys.*, M.I.T., v. 22, 1943, p. 115–135. 17.5 × 25.5 cm.

This paper provides the coefficients for the expansion of the nth derivative of $\phi(x)$, when $\phi(x)$ is expanded in terms of central differences, that is to say, for the coefficients of $\delta^{2\nu-1}$ and $\delta^{2\nu}$ in the expansions:

$$\omega^{2n-1}\phi^{(2n-1)}(x) = \sum_{\nu=n}^{\nu=k} A_{2\nu-1}^{2n-1}\delta^{2\nu-1} + R_1,$$

$$\omega^{2n}\phi^{(2n)}(x) = \sum_{\nu=n}^{\nu=k} A_{2\nu}^{2n}\delta^{2\nu} + R_2,$$

where R_1 and R_2 are remainders, whose explicit forms are found in standard treatises on finite differences. For the mean differences of odd order, we replace the customary notation, $\mu\delta^{2\nu-1}$ or $\Box\delta^{2\nu-1}$, by the simpler notation $\delta^{2\nu-1}$. The original expansion of $\phi(x)$ in terms of central differences is usually referred to as the Newton-Stirling series.

The author defines the coefficients in terms of $B_m^{(n)}(x)$, the Bernoulli polynomial of mth order and nth degree, as follows:

$$A_{2\nu-1}^{2n-1} = \frac{1}{(2\nu-2n)!}B_{2\nu-2n}^{(2\nu)}(\nu), \qquad \text{and} \qquad A_{2\nu}^{2n} = \frac{n}{\nu(2\nu-2n)!}B_{2\nu-2n}^{(2\nu)}(\nu).$$

It is unfortunate that he did not also state them in terms of the familiar differential coefficients of zero, namely,

$$A_{2\nu-1}^{2n-1} = D^{2n-1}0^{[2\nu]-1}/(2\nu-1)!, \qquad \text{and} \qquad A_{2\nu}^{2n} = D^{2n}0^{[2\nu]}/(2\nu)!.$$

Coefficients of even order can be immediately written down from those of odd order by the obvious relationship

$$A_{2\nu}^{2n} = (n/\nu)\, A_{2\nu-1}^{2n-1}.$$

The author used the following new identity in a partial check of his computations:

$$A_{2\nu-1}^{2n-1} = \frac{1}{2\nu-1}\left[(2n-1)A_{2\nu-2}^{2n-2} - (\nu-1)^2\frac{1}{2n}A_{2\nu-2}^{2n}\right].$$

In order to achieve his purpose in making "this table the 'ultimate' in coefficients for numerical differentiation," the author computes the coefficients as far as the 52nd derivative. For the first 30 derivatives the author gives the exact values, in common fractions, up to the difference of 30th order, and also for some coefficients of differences beyond the 30th. "For all derivatives beyond the 30th, exact values are given for coefficients of differences going as far as some difference between the 41st and 52nd. Elsewhere, that is, for most of the coefficients of the 31st to 42nd differences, 18 significant figures are given, with accuracy to within 0.6 unit in the last significant figure."

To put it otherwise, the derivative formulas can now be written to at least 18 significant figures for the first 32 derivatives up to differences of 42nd order, the 33rd, 34th, 35th, 36th, 37th, 38th, 39th, 40th, and 41st derivatives up to orders 43, 44, 45, 46, 47, 48, 49, 50, and 51 respectively, and the derivatives of orders from 42 to 52 to differences of 52nd order.

The coefficients were checked to about 10 significant figures by means of the recurrence formula given above.

Although it is difficult to see where these tables would be used in the calculation of derivatives of such high order, since differences beyond eight or ten are almost never encountered, nevertheless the coefficients themselves are related to some interesting functions and could quite conceivably be of importance in problems other than those for which the table has been computed. We are greatly indebted to the author for providing us with these new constants.

H. T. D.

150[L].—CARL AUGUST HEUMAN (1870–), "Tables of complete elliptic integrals," *J. Math. Physics*, M.I.T., v. 20, 1941, p. 127–206. 17.5 × 25.4 cm.

The first two tables in this paper give values to 6D of the complete elliptic integrals of the first and second kinds, $F_0(\alpha)$ and $E_0(\alpha)$, for $\alpha = 0°.0(0°.1)90°$, with first differences; except that the differences are omitted for $\alpha > 65°$ in the case of $F_0(\alpha)$. The functions $F_0(\alpha)$ and $E_0(\alpha)$ are $2/\pi$ times the Legendre integrals F and E. Linear interpolation is sufficiently accurate except in the range $\alpha > 65°$ for $F_0(\alpha)$. To fill this gap there is given a table of an auxiliary function in which linear interpolation can be used:

$$G_0(\alpha) = F_0(\alpha) + (2/\pi) \ln (90° - \alpha°)$$

for $\alpha = 65°.0(0°.1)90°$, with first differences.

An extensive table of complete elliptic integrals of the third kind $\Lambda_0(\alpha, \beta)$ is also given, p. 160–197, where

$$\Lambda_0(\alpha, \beta) = F_0(\alpha)E(\alpha', \beta) - \{F_0(\alpha) - E_0(\alpha)\}F(\alpha', \beta).$$

Here $\alpha' = \pi/2 - \alpha$ and $F(\alpha, \beta)$ and $E(\alpha, \beta)$ are the incomplete integrals of the first and second kinds. Tabular values are for $\alpha = 0°(1°)90°$, $\beta = 0°(1°)90°$, and for $\alpha = 0°.0(0°.1)5°.9$, $\beta = 80°(1°)89°$, all to 6D.

Heuman obtained his values of $F_0(\alpha)$ and $E_0(\alpha)$ directly from values of log F and log E in Table I of Legendre's *Exercices de calcul intégral sur divers ordres de transcendantes et sur les quadratures*, Paris, 1816, Tome 3, and *Traité des fonctions elliptiques*, Paris, 1826, Tome 2, with the aid of Vega's ten-place *Thesaurus Logarithmorum*. The values of $\Lambda_0(\alpha, \beta)$ were computed from Legendre's Table I, to 12D–14D, for each tenth of a degree, and Table IX, the latter being a table of the incomplete integrals $F(\alpha, \beta)$ and $E(\alpha, \beta)$ to 9D–10D, for each degree. A list of 42 errors in Legendre's tables is given, most of them being in the incomplete integrals.

An appendix to the paper contains applications to the motion of a spherical (or more general) pendulum, and the gyroscopic pendulum. In the former case a chart is given for the apsidal angle as function of the two independent variables z_1 and z_2, which are the greatest and least heights attained by the bob during its motion. The gyroscopic pendulum is considered in the case where the spindle starts at an inclination θ_0 with zero initial velocity. Two charts are given, for the apsidal and zonal angles respectively, as functions of the two independent variables θ_0 and μ where μ is the coefficient of stability.

P. W. KETCHUM

151[L].—MATHEMATICAL TABLES PROJECT, New York, A. N. LOWAN, technical director, *Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments*, New York, Columbia University Press, 1943, xliv, 403 p. 19.6 × 26.5 cm. Reproduced by a photo offset process. $5.00.

The main part of this useful volume consists of a table of real and imaginary parts of $J_0(z)$ and $J_1(z)$ as functions of $z = \rho e^{i\vartheta}$, for $\rho = [0.00(0.01)10; 10D]$, $\vartheta = 0°(5°)90°$. The last twenty pages contain a table of the five-point Lagrangean Interpolation Coefficients for $p = [0.000(0.001)1; 10D]$, twice previously published (*MTAC*, p. 94). There is also a short foreword by H. BATEMAN, and a long Introduction by Mr. LOWAN, followed by a Bibliography (65 items) of Tables of Bessel functions and of applications of Bessel functions of a complex argument of the form $ri^{\frac{1}{2}}$. The four pages of "Contour Lines" for $J_0(z) = J_0(\rho e^{i\phi}) = U(\rho, \phi) + iV(\rho, \phi), J_0(z) = J_0(\rho e^{i\phi}) = Re^{i\theta}, J_1(z) = J_1(\rho e^{i\phi}) = u(\rho, \phi) + iv(\rho, \phi), J_1(z) = J_1(\rho e^{i\phi}) = Se^{i\sigma}$, are of considerable interest. The publication of the Bessel function tabulation makes possible the numerical solution of a large number of problems, which have hitherto only been discussed symbolically. Apart from tables with pure imaginary arguments, and for arguments in the form $ri^{\frac{1}{2}}$, DINNIK's short tables[1] of $J_0(z)$ and $J_1(z)$ are the only ones previously published.

Bessel functions are ubiquitous in mathematical physics, from acoustics to stochastics. When the wave equation in more than one dimension is separated in rectangular coordinates, the separated equations have an irregular singular point at infinity, and the solutions are exponential or trigonometric functions. If the coordinate system involves a point center or a single axis of revolution, such as for spherical or circular cylindrical coordinates for instance, the radial equation has a regular singular point at the origin and an irregular singular point at infinity. The general solution of this type equation is the confluent hypergeometric function, but in the great majority of cases of practical interest, the special case of the Bessel equation is the result. There are other more complicated coordinate systems in which the wave equation separates, but the resulting ordinary differential equations have a more complicated system of singular points, and the solutions (Mathieu functions, Lamé functions and the like) are not well enough tabulated to make them readily usable in specific problems.

The peculiar position of the Bessel function is again evident when one studies the applications of the Laplace transform, whether in wave or diffusion problems or in probability theory. In many coordinate systems the characteristic integral expression

$$\int e^{iz\cos\phi + in\phi}d\phi$$

occurs. This can be shown to be proportional to one or another of the solutions of the Bessel equation of order n, depending on the choice of the contour for the integral.

Perhaps part of the reason for the preeminence of the Bessel function in mathematical physics is that it is neither too complex nor too simple. A solution of the wave equation expressed in terms of trigonometric functions would not be adequate to use in studying the behavior of waves from point or line sources, which are best expressed in terms of Bessel functions. On the other hand, although a solution of the oblate spheroidal waves from a circular disk would be even more valuable than a solution for a point source, the oblate spheroidal functions needed for such a solution are so much more difficult to compute than are the Bessel functions, that no adequate tables are yet available from which to obtain complete numerical results. Bessel function solutions are complex enough to exhibit certain general wave properties, and yet simple enough to tempt the computer.

As the Bibliography well suggests, tables of Bessel functions for real and for pure imaginary arguments are now fairly numerous. A great number of problems can be solved numerically by their use, and the tables under review do not represent an advance in this direction. There are a number of important problems, however, whose solution requires values of the Bessel functions for complex values of the argument; and in this field the present *Table* represents a distinct advance.

Uses for the functions, for the argument $\rho\sqrt{i}$, are fairly well known (the ber, bei and related functions), and a number of tables have been published of these functions.[2] Problems relating to the behavior of electromagnetic waves in cylindrical conductors (skin-effect) require the use of these functions, as also do calculations of the fluctuation of temperature in a cylinder whose surface temperature is a sinusoidal function of time (furnace with

thermostatic control). The present tables are more complete than any of the earlier ones of ber x, bei x (the B.A.A.S. Tables, for instance) for this special case of $\vartheta = 45°$.

For other values of ϑ, however, the present tables are unique, and their publication makes possible calculations of considerable immediate interest. The behavior of microwaves in cylinders of dielectric material in which there is energy dissipation is one application; and the scattering of electromagnetic waves from such a cylinder is another. The attenuation of acoustic waves down hollow cylinders[3] can now be calculated, and computing the transmission of sound in a shallow sea with absorbing bottom is made considerably easier. In fact, wherever cylindrical waves cause energy dissipation, either in the medium or at the boundary, Bessel functions of complex argument are encountered, and the present tables are of use. The companion volume, giving values of N_0 and N_1 for complex values of the argument, will also be most useful when it is published.

The reviewer has not had the leisure to test the accuracy of the present Tables. However, the accuracy of the previous publications of the Mathematical Tables Project is well known.

PHILIP M. MORSE

Massachusetts Institute of Technology

[1] A. N. DINNIK, All Russian Central Committee, [Communications], 1922, p. 121–126; reprinted in K. HAYASHI, *Fünfstellige Funktionentafeln* . . . , Berlin, Springer, 1930, p. 105–109.

[2] A. G. WEBSTER, B.A.A.S., *Report*, 1912, p. 56–68; ber x and bei x and their derivatives, $x = [0.0(0.1)10.0; 9D]$, with 7 differences. For corrections by H. G. SAVIDGE see B.A.A.S., *Report*, 1916, p. 122; there are also corrections by H. B. DWIGHT. SAVIDGE gave tables of ker, kei and their first derivatives and other tables in B.A.A.S., *Reports*, 1915, p. 36–38, and 1916, p. 108–121. An abridgement of some of the tables in these Reports is given by H. B. DWIGHT in (a) *Tables of Integrals and Other Mathematical Data*, New York, Macmillan, 1934; see RMT **154**; and (b) *Mathematical Tables*, New York, McGraw-Hill, 1941; see RMT **143**. The latter has 5-place tables of functions and derivatives for $\vartheta = 45°, 135°$.

[3] P. M. MORSE, "The transmission of sound inside pipes," Acoustical So. Am., *J.*, v. 11, 1939, p. 205.

152[L].—E. O. POWELL, "An integral related to the radiation integrals," *Phil. Mag.*, s. 7, v. 34, Sept. 1943, p. 600–607. 17 × 25.3 cm.

The following function is tabulated

$$Rl(x) = \int_1^x \ln y dy/(y-1) = \sum_1^\infty (-1)^{n+1}(x-1)^n n^{-2} \qquad (0 \leqslant x \leqslant 2),$$

$$= \ln x \ln(1-x) - \pi^2/6 + \sum_1^\infty x^n n^{-2} \qquad (0 \leqslant x \leqslant 1);$$

$Rl(x) + Rl(1/x) = 1/2(\log x)^2$. From these relations the values of $Rl(x)$ were computed for $x = [0.00(0.01)2.00(0.02)6.00; 7D]$, with second central differences, negative throughout. The author writes as follows: "It is hoped that the tabular values will be found correct to six decimal places; the seventh should not usually be in error by more than one unit." The "differences are sufficient to give seven-figure accuracy when x is greater than 0.2, six-figure between 0.1 and 0.2; below this limit they increase very rapidly."

The integral $\int_1^x \ln y dy/(y-1)$ was recently encountered in a physical problem.[1] This integral is equal to $Ri_{-1}[\ln(1/x)] - \pi^2/6$, where $Ri_n(x) = \int_x^\infty dy/y^n(e^y - 1)$, an integral tabulated by AIREY for 6 values of n (see *MTAC*, p. 140, no. 46). DEBYE gave[2] a short table of $(1/x)\int_0^x y dy/(e^y - 1) = \pi^2/(6x) - (1/x)Ri_{-1}(x)$.

[1] S. R. FINN and E. O. POWELL, "The chemical and physical investigation of germicidal aerosols. II: The aerosol centrifuge," *J. Hygiene*, v. 42, 1942, p. 364.

[2] P. DEBYE, "Interferenz von Röntgenstrahlen und Wärmebewegung," *Annalen d. Phys.*, s. 4, v. 43, 1914, p. 85–86.

153[L].—F. Vandrey, "Tafel der acht ersten Kugelfunktionen zweiter Art," *Z. angew. Math. Mech.*, v. 20, 1940, p. 277–279. 20.8 × 29.5 cm.

These tables emanated from the aerodynamic research laboratory at Göttingen. They present the values of the Legendre spherical functions of the second kind $Q_n(x)$, from $Q_0(x)$ to $Q_7(x)$, for $x = [0.00(0.01)1.00;\ 5D]$. By virtue of the relation $Q_n(-x) = (-1)^{n+1}Q_n(x)$ the table covers also the range $-1 \leqq x \leqq 0$.

In calculating the table the values of the function

$$Q_0(x) = \tanh^{-1} x = \tfrac{1}{2} \ln [(1+x)/(1-x)]$$

were taken from the 9-place table of K. Hayashi, *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen*, Berlin, Springer, 1926, p. 9–87. $Q_1(x)$ to $Q_4(x)$ were calculated from the following equations: $Q_1(x) = xQ_0(x) - 1$, $Q_2(x) = P_2(x)Q_0(x) - (3/2)x$, $Q_3(x) = P_3(x)Q_0(x) - (5/2)x^2 + \tfrac{2}{3}$, $Q_4(x) = P_4(x)Q_0(x) - (35/8)x^3 + (55/24)x$. Q_5 to Q_7 were then determined by means of the recurrence formula,

$$nQ_n(x) + (n-1)Q_{n-2}(x) - (2n-1)xQ_{n-1}(x) = 0.$$

In the tables a dot placed after the upper part of the fifth figure indicates that this figure has been increased. Linear interpolation to 3 places is possible in $Q_0(x)$ to $Q_6(x)$ from 0 to about 0.86, and in $Q_7(x)$ from 0 to 0.69.

154[L, M].—H. B. Dwight, *Tables of Integrals and other Mathematical Data*, New York, Macmillan, 1934, x, 222 p. 14 × 21.5 cm. $1.75. See also RMT **30**.

The first important American table of integrals was the one by the brilliant Harvard physicist, B. O. Peirce (1854–1914) which started in a small way as a 32-page pamphlet (1889), and was bound in with the second edition of W. E. Byerly's *Elements of the Integral Calculus*, Boston, 1889. After tremendous labor this, *A Short Table of Integrals*, was expanded to a book of 144 pages (Boston, 1910). The third edition (156 p.), revised by W. F. Osgood, appeared in 1929. After 479 indefinite integrals, there are 44 miscellaneous definite integrals, followed by more than 60 formulae under the heading Elliptic Integrals. Then follow (p. 73–115, nos. 570–938) various auxiliary formulae in trigonometric, hyperbolic, elliptic, and Bessel functions; series; derivatives; Green's theorem and allied formulae; table of mathematical constants; general formulae of integration; note on interpolation. The final pages (116–154) contain various tables including those of the probability integral, of elliptic integrals, of hyperbolic functions, and of $\Gamma(n)$.

And now we are considering a volume by a professor of electrical engineering at Massachusetts Institute of Technology. In a general way the volume of Dwight is not greatly dissimilar to that of Peirce but it is considerably more elaborate. Dwight has much more detailed numbering of formulae[1] (with many cross references), and careful statements of ranges for which they are valid. On p. 219–220 are listed 35 volumes of tables, treatises, etc., to which references are often given as authorities for statements made, or as sources where more elaborate results appear. Peirce's tables in this list are naturally referred to several times. Tables 1050, of ber, bei, ber′, bei′, ker, kei, ker′, kei′, are taken from those of Webster and Savidge in B.A.A.S. *Reports*, 1912, p. 56–68; 1915, p. 36–38; 1916, p. 122. Dwight has corrected the value for bei 8.9 given by Webster as $-8.002\cdots$ instead of $-28.002\cdots$. Dwight made this correction first in Amer. Inst. Electr. Engs., *Jn.* v. 42, 1923, p. 830. But he failed to correct Webster's value for bei′ 3.7 0.134 686 760 which should have been 0.131 486 760 (Dwight, A.I.E.E., *Trans.*, v. 58, 1939, p. 787). On pages 215–217 of Table 1050 are original tables of $\mathrm{ber}_n\, x$, $\mathrm{bei}_n\, x$, $\mathrm{ber}'_n\, x$, $\mathrm{bei}'_n\, x$, $\mathrm{ker}_n\, x$, $\mathrm{kei}_n\, x$, $\mathrm{ker}'_n\, x$, $\mathrm{kei}'_n\, x$, for $n = 1(1)5$; $\mathrm{ber}_n\, x + i\, \mathrm{bei}_n\, x = J_n(xi^{\frac{3}{2}}) = i^n I_n(xi^{\frac{1}{2}})$; $\mathrm{ker}_n\, x + i\, \mathrm{kei}_n\, x = i^{-n}K_n(xi^{\frac{1}{2}})$. The first four of these tables, 3D–6D, were first published in A.I.E.E., *Trans.*, v. 42, 1923, p. 858, and reprinted in *Trans.*, v. 48, 1929, p. 814–815; of the second four,

mostly 6D–8D, the greater part of the values were calculated by W. H. HASTINGS as thesis work at M.I.T. in 1927.

The section on Bessel Functions, nos. 800–845, is a very useful collection of recurrence formulae, identities (functions and integrals), with real and complex arguments. The notation of Gray and Mathews and Macrobert's *A Treatise on Bessel Functions*, London, 1922 (1931), is here employed.

The section of formulae on Elliptic Functions and Integrals occupies nos. 750–789.2 and in nos. 1040–1041 there are two brief tables of K and E. In K, 87°.6, *for* 4.562, *read* 4.561.

The series and formulae nos. 1–50 include some dealing with Bernoulli and Euler numbers.

This volume, by the author of several mathematical tables, is an excellent and exceedingly useful one, compiled with great care. The "Contents" and "Index" are wholly adequate. Some other errata are noted in MTE **32**.

R. C. A.

[1] We have already drawn attention (*MTAC*, p. 66) to a valuable volume, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, 1922 (corrected reprint 1939), 314 p. The Mathematical Formulae (p. 1–219) were prepared by the physicist EDWIN P. ADAMS. He writes, "In order to keep the volume within reasonable bounds, no tables of indefinite and definite integrals have been included. For a brief collection, that of the late Professor B. O. Peirce can hardly be improved upon; and the elaborate collection of Bierens de Haan show how inadequate any brief tables of definite integrals would be. A short list of useful tables of this kind, as well as of other volumes, having an object similar to this one, is appended." Nevertheless, there is much in common with the volumes under review. Dwight doubtless received more than one suggestion from this volume which has a goodly number of literature references. In the useful chapter on infinite series, p. 109–144, there is still one error, as Mr. W. D. Lambert, of the Coast and Geodetic Survey, has recently pointed out; on p. 122, under 6.42, no. 4, the third and fifth terms of the right-hand member should each be preceded by the sign —. Legendre and Bessel functions are considered at some length on p. 191–219.

155[L, M].—A. H. HEATLEY, "A short table of the Toronto function," R. So. Canada, *Trans.*, v. 37, sect. III, 1943, p. 13–29. 16.2 × 25 cm.

By "Toronto function" the author designates the three-parameter function $T(m, n, r)$ defined in terms of the four-parameter function

$$T(m, n, p, a) = \int_0^\infty t^{m-n} e^{-p^2t^2} I_n(2at)dt$$

by means of the relation

$$T(m, n, r) = 2r^{n-m+1} e^{-r^2} p^{m-n+1} T(m, n, p, a),$$

where $r = a/p$. Many relations between the Toronto function and other functions such as the confluent hypergeometric function $M(\alpha, \gamma, x)$, Bessel functions of fractional orders, the error function, etc., are summarized in Table 1-A. The author gives also (without derivation) a number of recurrence formulae between three T's whose parameters m and n differ by integers, as well as a number of formulae for special values of these parameters. He also gives the differential equation satisfied by T when considered as a function of r and an asymptotic expansion based on the confluent hypergeometric function.

The short table included with the article gives 5-place values of $T(m, n, r)$ for the following values of the parameters:

$$\begin{aligned} m &= -\tfrac{1}{2}; & n &= -2(0.5)2; & r &= 0(0.2)2;\ 5,\ 6,\ 10,\ 25,\ 50,\\ m &= 0; & n &= -2(0.5)2; & r &= 0(0.2)4;\ 5,\ 6,\ 10,\ 25,\ 50,\\ m &= \tfrac{1}{2}; & n &= -2(0.5)2; & r &= 0(0.2)2;\ 4,\ 5,\ 6,\ 10,\ 25,\ 50,\\ m &= 1; & n &= -2(0.5)2; & r &= 0(0.1)2(0.2)3. \end{aligned}$$

The computations of $T(m, n, r)$ were carried out on the basis of the relations in Table 1-A for sufficiently small values of r and on the basis of the asymptotic expansion for sufficiently large r (mostly for $r > 4$).

The computed values were checked with the aid of the recurrence formulae and occasional differencing. In order to compute $T(m, n, r)$ for small values of r, the author found it necessary to prepare two small tables of the function $M(\alpha, \gamma, x)$ and $e^{-x}M(\alpha, \gamma, x)$. The first lists $M(\frac{1}{4}, 3, x)$, $M(\frac{1}{4}, 2, x)$, $M(\frac{1}{4}, 1, x)$, $M(\frac{3}{4}, 2, x)$ and $M(\frac{5}{4}, 3, x)$ for 11 values of x ranging from 0 to 4 at irregular intervals. The second lists $e^{-x}M(\frac{7}{4}, 3, x)$, $e^{-x}M(\frac{7}{4}, 2, x)$, $e^{-x}M(\frac{3}{4}, 1, x)$, $e^{-x}M(\frac{5}{4}, 2, x)$ and $e^{-x}M(\frac{11}{4}, 3, x)$ for x ranging between 0 and 4 at the same intervals. There is a gap in the values of $T(-\frac{1}{2}, n, r)$ between $r = 2$ and $r = 5$ and a gap in the values of $T(\frac{1}{2}, n, r)$ between $r = 2$ and $r = 4$.

The reviewer wonders whether it would not have been more expedient to base the computation of $T(m, n, r)$ for small values of r directly on the differential equation rather than on its expression in terms of the confluent hypergeometric functions whose series expansion is so slow that the author had to take 19 terms of the series.

Conceivably the method developed by C. Lanczos ("Trigonometric interpolation of empirical and analytical functions," *J. Math. Physics*, M.I.T., v. 17, 1938, p. 123–199) might have yielded expressions valid in a range of r including the gaps above noted.

Arnold N. Lowan

156[L, S].—L. D. Rosenberg, "Metod rascheta zvukovykh poleĭ, obrazovannykh raspredelennymi sistemami izluchateleĭ, rabotai͡ushchimi v zakrytykh pomeshchenii͡akh" [Method of calculating sound fields generated by distributed systems of radiators operating in closed places] *Zhurnal Tekhnicheskoĭ Fiziki*, v. 12, 1942, p. 247–248. 16.6 × 25.9 cm.

These pages at the end of the article contain a table of $T(u, \alpha) = \int_\alpha^{\frac{1}{2}\pi+\alpha} e^{-u \sin x} dx$, for $\alpha = [0.0(0.1)0.7, 0.785; 5D]$, $u = 0.0(0.1)1.0$, 1.5, 2.0, 3.0, 3.5, 4, 5, 6, 8, 10, 15, 20, 30, 40, 50, 70, 100.

157[M].—A. N. Lowan and J. Laderman, "Table of Fourier coefficients," *J. Math. Phys.*, M.I.T., v. 22, 1943, pp. 136–147. 17.5 × 25.5 cm.

In this paper we find the tabulated values of the two functions

$$S(k, n) = \int_0^1 x^k \sin n\pi x \, dx, \qquad \text{and} \qquad C(k, n) = \int_0^1 x^k \cos n\pi x \, dx,$$

over the range $n = 1(1)100$, and $k = 0(1)10$. The tables are given to 10 decimal places with a plus sign indicating that the eleventh decimal place is 5 or larger. This is a rather interesting innovation, since the usual rule is to provide in the last place a figure which is not in error by more than .5 units, the sign of the error being either plus or minus. In this connection it might be worth while calling attention to the device employed by L. M. Milne-Thomson and L. J. Comrie in their *Standard Four-Figure Mathematical Tables*, London, 1931, who say with respect to their tables that "a further indication of the figures omitted is given by placing a 'high' dot after the last figure if these omitted figures, in units of the last figure retained, lie between .1666... (i.e. $\frac{1}{6}$) and .5000..., and a 'low' dot if they lie between .5000... and .8333... (i.e. $\frac{5}{6}$)."

The two functions tabulated satisfy the recurrence formulas:

$$S(k, n) = S(1, n) - \frac{k(k-1)}{n^2\pi^2} S(k-2, n), \qquad C(k, n) = \tfrac{1}{2}k\, C(2, n) - \frac{k(k-1)}{n^2\pi^2} C(k-2, n);$$

and the "cross relations":

$$S(k, n) = S(1, n) - \frac{k}{n\pi} C(k-1, n), \qquad C(k, n) = -\frac{k}{n\pi} S(k-1, n).$$

As to the accuracy of the table we may quote the authors, who say: "The computations were performed originally to twenty decimal places with such extreme care, that the

'cross relation' checks on the twenty-place values did not reveal a single error. The present table is an abridged version of the original twenty-place manuscript."

The table is designed to facilitate the computation of Fourier coefficients for functions which may be represented by polynomials of degrees not exceeding 10.

Since the table under review is unique in the field of harmonic analysis, which has been developed vigorously during the past forty years by astronomers, physicists, meteorologists, and economists, it may be of interest to indicate its relationship to other tables.

In harmonic analysis the interest has focused upon the two sums,

$$A_n = \frac{2}{N}\sum_{t=1}^{N} f_t \cos\frac{2n\pi t}{N}, \qquad \text{and} \qquad B_n = \frac{2}{N}\sum_{t=1}^{N} f_t \sin\frac{2n\pi t}{N};$$

or the corresponding integrals,

$$A_n' = \frac{2}{a}\int_0^a f(t)\cos\frac{2n\pi t}{a}\,dt, \qquad \text{and} \qquad B_n' = \frac{2}{a}\int_0^a f(t)\sin\frac{2n\pi t}{a}\,dt,$$

which are usually to be evaluated by finite integration.

Hence most tables designed for use in harmonic analysis provide values of the functions

$$M\cos\frac{2n\pi}{p} \qquad \text{and} \qquad M\sin\frac{2n\pi}{p}.$$

The following is a bibliography of such tables:

H. H. TURNER, *Tables for facilitating the use of harmonic analysis*. London, Milford, 1913, 46 p.

L. ZIPPERER, *Tafeln zur harmonischen Analyse periodischer Kurven*, Berlin, Springer, 1922, 12 p.

L. W. POLLAK, *Rechentafeln zur harmonischen Analyse*, Leipzig, Barth, 1926, 23 + 17 + 240 p.

L. W. POLLAK, *Handweiser zur harmonischen Analyse*, Prague, Bursík & Kohout, 1928, 72 + 98 p.

P. TEREBESI, *Rechenschablonen für harmonische Analyse und Synthese*. Berlin, Springer, 1930, 13 + 11 p. + 12 tables.

A. HUSSMANN, *Rechnerische Verfahren zur harmonischen Analyse und Synthese, mit Schablonen für eine Rechnung mit 12, 24, 36, oder 72 Ordinaten*. Berlin, Springer, 1938. 28 p. + 12 folding tables.

K. STUMPFF, *Tafeln und Aufgaben zur harmonischen Analyse und Periodogramrechnung*. Berlin, Springer, 1939, vii + 174 p.

Since the demands of practical statistics do not usually require many decimal places, the above tables are computed mainly to three or four significant figures. Thus the last table, which is typical, contains three-place values of the two functions for $M = 1$, $p = 3(2)39$, $n = 0(1)\frac{1}{2}p$; $p = 2(4)40$, $n = 0(1)\frac{1}{2}p$; $p = 4(4)40$, $n = 0(1)\frac{1}{4}p$; $M = 0(1)1000$, $n = 1$, $p = 8$, 12, 16, and 24; $M = 2(2)100$, $p = 21(1)27$, 29, $30(1)35$, $37(1)39$, $n = 0(1)\frac{1}{2}p$; $M = 1(1)100$, $p = 20(4)40$, $n = 0(1)\frac{1}{4}p$.

Since occasionally greater accuracy is required than is provided by these tables the reviewer has prepared in manuscript form an 8-place table of the two functions for $M = 1$, $p = 5(1)75$, $n = 0(1)p$.

It will be clear that the table which is the subject of this review will be useful in harmonic analysis in those instances when the statistical function $f(t)$ can be approximated by a polynomial of degree not exceeding ten. The values of A' and B' are then immediately approximated by appropriate sums of the the tabulated values of $C(k, n)$ and $S(k, n)$, when we use the transformation $t = ax$. The table was computed during 1940–41 in connection with work of the MATHEMATICAL TABLES PROJECT.

H. T. D.

MATHEMATICAL TABLES—ERRATA

References have been made to errata in RMT **143** (DWIGHT), **145** (KO and WANG), **150** (LEGENDRE), **154** (ADAMS, DWIGHT, WEBSTER), N**18** (SILBERSTEIN), N**19** (CARRINGTON, JAHNKE & EMDE), and N**20** (BOYS, BYRNE, KNAPPEN).

29. BESSEL and HANSEN Tables of Bessel Functions.

It is well known that Bessel published in 1826 a table of $J_n(x)$, $n = 0, 1$; $x = [0.00(0.01)3.20; 10D]$. Because of misinformation published in sources usually highly authoritative it is not so well known that one of P. A. Hansen's tables of 1843 was of $J_n(x)$, $n = 0, 1$; $x = [0.0(0.1)20; 6D]$. Among others this was reprinted by Lommel in 1868.

In the absurd paragraph of 16 lines on "Tables de fonctions de Bessel" in *Encyclopédie des Sciences Mathématiques*, II, 5, 2, p. 228, one reads (omitting footnote references) "F. W. Bessel a calculé des tables des fonctions J_0 et J_1 pour des arguments variant de $x = 0$ à $x = 3, 2$. P. A. Hansen les a étendues jusqu'à $x = 10$, E. Lommel jusqu'à $x = 20$." [Hansen did not extend Bessel's 10-place table for hundredths of a unit to $x = 10$, and Lommel did not extend Hansen's table.] In Watson's *Bessel Functions*, 1922, p. 655, there are three misleading statements, "Hansen constructed a Table of $J_0(x)$ and $J_1(x)$ to six places of decimals with a range from $x = 0$ to $x = 10.0$ with interval 0.1 . . . Hansen's table was reprinted . . . by Lommel who extended it to $x = 20$." E. M. Horsburgh repeats one misleading phrase and makes one new error, on p. 57 of his *Modern Instruments and Methods of Calculation*, 1914, "P. A. Hansen's extension of Bessel's table is reproduced by . . . E. Lommel. . . . It gives $J_0(x)$ and $J_1(x)$ from $x = 0$ to $x = 20$ at intervals of .01 throughout the lower part of the range." It would seem as if all three authors did not recognize that $J_n(x) = I_{2x}^n$ (Hansen) $= J^n(x)$ (Lommel) [$= I_x^n$ (Bessel)].

R. C. A.

30. E. W. BROWN with the assistance of H. B. HEDRICK, *Tables of the Motion of the Moon*, New Haven and London, 1919. Compare *MTAC*, p. 29.

Apart from the erratum on p. xiv of these *Tables*, two lists of errata were published by BROWN, namely: (1) in *Astr. J.*, v. 34, 1922, p. 54; and (2) in Yale Univ., Astr. Observatory, *Trans.*, v. 3, 1926, p. 157; in l. 4 from the bottom, *for* 92-l′, *read* 82-l′. Eight other errors sent to us are now presented. The one in Section II was supplied by D. H. SADLER, Superintendent of H. M. Nautical Almanac Office, three of those in Sections I and VI came from W. J. ECKERT, Director of the Nautical Almanac Office, Washington, and the rest, in Sections I, III, VI, were taken by DIRK BROUWER, Director of the Yale University Observatory, from two copies of the *Tables* used by the late E. W. BROWN.

Section I

P. 44, l. 3, *for* 70, *read* 71.
80, l.-2, *for* $n^{-\frac{1}{2}}$, *read* $n^{+\frac{1}{2}}$.
109, l. 10, *for* log cos ω, *read* log sin ω.

Section II

P. 31, arg. 84 for 1966, *for* 2305, *read* 2405.

Section III

P. 160, col. 17, arg. 16.5, *for* 19608, *read* 19598.
P. 187, col. 153, arg. 9.0, *for* 23878, *read* 23868.

Section VI

P. 63, col. 47, arg. 180, 190, 200, *for* 110, 117, 112, *read* 210, 217, 212.
P. 92, 0°59′0″, *for* 007 4464, *read* 007 4454.

31. Edwin Chappell (1883–1938), *A Table of Coefficients to Facilitate Interpolation by Means of the Formulæ of Gauss, Bessel and Everett.* London, The Author, 1929, viii, 27 p. 22.3 × 28 cm.

As the author of these tables is no longer living, it may be of interest to mention that the balance of the 110 copies printed (by Chappell himself) are now available from Scientific Computing Service Ltd., 23 Bedford Square, W.C. 1.

Besides the two corrections given by the author on page 2, the formula on the second page of the Introduction for $G^{vi} = B^{vi}$ should be preceded by a minus sign. On pages 26 and 27 the values of $G^{vi} = B^{vi}$ for 0.38 and 0.62 (which are the same) are both in error: *for* 0.004 56167, *read* 0.004 56157. The occurrence of this error (which was found about ten years ago by Mr. D. H. Sadler) in two places shows that it is an error in recording the original calculations, rather than one of composition. The fact that the values in Table I for 0.380 and 0.620 are correct shows that a comparison between the two tables was not made.

This table was compiled at the suggestion of the present writer, whose copy is inscribed "3/110. With the author's compliments and in recognition of the suggestion which has resulted in this book." The calculations were done on the first Brunsviga Dupla calculating machine imported into England—a machine which, by curious fate, also came into my hands some years after Chappell's death.

L. J. C.

Editorial Note: This well printed and bound little volume of Chappell is worthy of a place in any large mathematical library, along with his *Five-figure Mathematical Tables: consisting of logs and cologs of numbers from* 1 *to* 40,000, *illogs (antilogs) of numbers from* .0000 *to* .9999, *lologs (logs of logs) of numbers from* 0.00100 *to* 1,000, *illologs (antilologs) of numbers from* 6.0 *to* 0.5000, *Together with an Explanatory Introduction and Numerous Examples; also, trigonometrical functions and their logs of angles from* 0°–90° *at intervals of* 1 *minute, With Subsidiary Tables.* Edinburgh, London, Chambers, and New York, Van Nostrand, 1915, xvi, 320 p. Compare Q **4**. Chappell was much interested in Pepysiana, and published (1933–36) at least four items concerning Samuel Pepys and his bibliography.

32. H. B. Dwight, *Tables of Integrals and Other Mathematical Data*, New York, 1934. See RMT **154**. Most of the following errata were contributed by the author.

P. vii, *for* page numbers 113, 114, 114, 117, 118, 121, *read* respectively: 111, 112, 112, 115, 116, 119.

93.1. Change $3aX$ to $3a^2X$.

After the sub-headings on pages 101, 102 and 144, insert $(a > 0)$.

Omit No. 591.

603.2 For the part in square brackets, *read* $[\sin x \neq 0]$; and *for* logsin x, *read* $\log|\sin x|$.

603.4. For the part in square brackets, *read* $[\cos x \neq 0]$; and *for* logcos x, *read* $\log|\cos x|$. (See P. Franklin, *Differential Equations*, New York, 1933, p. 275).

P. 127, last line, *for* $\pi/4$, *read* π.

630.2, last line, *read* $[x^2 < \pi^2/4]$.

673.19, *for* 657.6 *read* 657.8.

679.19, *for* 657.5, *read* 657.7.

Since a statement is made in 810.8 about the size of the error from using the asymptotic series, it would be better to add $(1/\pi)K_0(x)$ to 810.6; and $(K_n(x)/\pi)e^{i(n+\frac{1}{2})\pi}$ to 810.7.

815.2 *for* $s = 1$, *read* $s = 0$.

836.4 *for* ker x, *read* ker$'x$.

850.1 *for* It, *read* The integral.

851.4. Change to the asymptotic series:

$$\Gamma(n+1) = n^{n+\frac{1}{2}}e^{-n}(2\pi)^{\frac{1}{2}}\left[1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \cdots\right].$$

Omit No. 861.9.

Table 1030: *read* $\log_{10}$ cosh .32 = .02187; sinh 1.20 = 1.5095; tanh 3.15 = .99633; tanh 3.55 = .99835.

Table 1050: *read* bei′ 3.7 = + 0.131 486 760. From x = 4.6 to 6.4, delete the last 3 digits of the values of bei′x.

P. 214, last line, *for* 109, *read* 122.

33. C. C. FARR, "On some expressions for the radial and axial components of the magnetic force in the interior of solenoids of circular cross-section," R. So. London, *Proc.*, v. 64, 1899, table p. 199–202. Reprinted without change in all five editions of JAHNKE and EMDE, 1909–1943.

On comparing with our recently calculated table for $dP_n(\cos\theta)/d\theta$, $n = 1(1)7$, we found 53 last figure errors, of which the 12 of more than a unit in the last figure are as follows: $n = 3$, 31°, *for* − 2.0654, *read* − 2.0656; $n = 4$, 5°, *for* − 0.8570, *read* − 0.8567; $n = 4$, 64°, *for* 1.6306, *read* 1.6300; $n = 5$, 54°, *for* 2.0173, *read* 2.0178; $n = 5$, 64°, *for* 1.5420, *read* 1.5418; $n = 5$, 67°, *for* 1.1187, *read* 1.1183; $n = 5$, 83°, *for* − 1.4831, *read* − 1.4827; $n = 6$, 23°, *for* − 3.0917, *read* − 3.0902; $n = 6$, 48°, *for* 2.3715, *read* 2.3713; $n = 6$, 57°, *for* 1.1936, *read* 1.1934; $n = 6$, 83°, *for* − 1.4487, *read* − 1.4484; $n = 7$, 70°, *for* − 1.984, *read* − 1.934. None of these 53 errors occur in the corresponding table of H. Tallqvist, *Grunderna af Teorin för Sferiska Funktioner jämte Användningar inom Fysiken*, Helsingsfors, 1905, p. 422–431.

We checked also the table of $P_n(\cos\theta)$, $n = 1(1)7$, $\theta = 0°(1°)90°$, on p. 121–123 of the fifth edition of JAHNKE and EMDE, and found no error.

MATHEMATICAL TABLES PROJECT

New York City

EDITORIAL NOTE.—The table of $P_n(\cos\theta)$ here referred to, is the one by HOLLAND, JONES and LAMB, originally with 92 errors (*MTAC*, p. 136–137). These were faithfully reproduced in JAHNKE & EMDE 1; but EMDE eliminated the errors, pointed out by TALLQVIST, in editions 2–5.

34. GREENHILL's first zero of $J_{-1/3}(x)$.

In Cambridge Phil. So., *Proc.*, v. 4, 1881, p. 68, G. GREENHILL gives the first positive root of $J_{-\frac{1}{3}}(x) = 0$ as 1.88; this was reproduced in the first edition (1909) of JAHNKE and EMDE, *Tables of Functions*, p. 106, and in A. DINNIK, (a) Russkoe fiziko-khimicheskoe Obshchestvo, *Zhurnal, Chast' fizicheskaia* v. 42, 1911, p. 4 of "Tablitsy funktsiĭ Besselia $J_{\pm\frac{1}{3}}$ i $J_{\pm\frac{2}{3}}$"; (b) *Archiv Math. Phys.*, s. 3, v. 18, 1911, p. 338.

J. R. AIREY gave the more accurate value 1.8663, in *Phil. Mag.*, s. 6, v. 41, 1921, p. 203. Y. IKEDA in *Z. angew. Math. u. Mech.*, v. 5, 1925, p. 81 derived the value 1.86635 0858. In a table of the first five zeros of $J_{\pm\frac{1}{3}}$ and $J_{\pm\frac{2}{3}}$, Dinnik gave the value 1.860 (Akad. Nauk, U.R.S.R., Kiev, *Prirodnicho-tekhnichniĭ viddil*, 1932, p. 12). D. H. L. writes, "more accurately this zero is 1.86635 08588 73895 . . . ."

R. C. A.

35. K. HAYASHI, *A. Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen und deren Produkte sowie der Gammafunktion* . . . , Berlin, 1926, table of $\sin^{-1} x$ for radian argument x = [0.00000(0.00001)0.001; 20D], [0.0010(0.0001)0.0999; 10D] and [0.100(0.001)1; 7D], p. 5–86; *B. Fünfstellige Funktionentafeln* . . . , Berlin, 1930, table of $\sin^{-1} x$, for x = [0.00(0.01)1; 5D], p. 2–4.

With the exception of the 100 values given to 20D all of Hayashi's inverse sines were proof-read against the Project's manuscript of the same function, see UMT **7**, p. 94–95, and the following errors were discovered:

A

x	*for*	*read*	*x*	*for*	*read*
.0132	.01320 03883	.01320 03834	.479	.4995154	.4995152
.0168	.01690 07904	.01680 07904	.481	.5017937	.5017950
.0257	.02570 28099	.02570 28299	.528	.5562452	.5562438
.0532	.05322 51278	.05322 51268	.543	.5744904	.5740056
.0558	.05585 89975	.05582 89975	.548	.5798714	.5799714
.0582	.05823 29154	.05823 29064	.589	.6298207	.6298209
.0626	.06264 09575	.06264 09580	.640	.6944985	.6944983
.0627	.06276 11548	.06274 11548	.657	.7171234	.7168325
.0660	.06604 80057	.06604 80102	.676	.7422209	.7423209
.0691	.06915 51094	.06915 51084	.683	.7519620	.7518620
.0693	.06935 56180	.06935 55890	.690	.7614939	.7614891
.0732	.07326 53661	.07326 55287	.696	.7698129	.7698116
.0759	.07597 29638	.07597 30638	.705	.7824131	.7824231
.0886	.08871 63294	.08871 63291	.707	.7852467	.7852472
.0891	.08921 83127	.08921 83145	.732	.8212534	.8212529
.132	.1323913	.1323864	.744	.8390418	.8390370
.152	.1525922	.1525915	.776	.8872990	.8882990
.198	.1993227	.1993171	.837	.9922139	.9917776
.267	.2702835	.2702787	.856	1.0274823	1.0274821
.387	.3973754	.3973758	.863	1.0411481	1.0411781
.390	.3906316	.4006316	.914	1.1530364	1.1530362
.408	.4202672	.4202624	.927	1.1863332	1.1863334
.457	.4746204	.4746195	.933	1.2026605	1.2026609
.467	.4858971	.4858950	.948	1.2473617	1.2468920
.470	.4891804	.4892908	.951	1.2564570	1.2564542
.472	.4915880	.4915580			

In addition to the above, errors of a unit in the last place (a) a unit too large, (b) a unit too small (including doubtful rounding off), occur at the following values of x:

(a) $x = .0015$, .0109, .0134, .0138, .0177, .0238, .0381, .0389, .0402, .0539, .0647, .0666, .0746–.0748, .0817, .0819, .0924, .0963–.0969, .304, .329, .359, .422, .468, .506, .531, .539, .567, .595, .628, .691, .694, .703 and .742.

(b) $x = .0209$, .0272–.0279, .0305, .0512, .0614, .0616, .0619, .0707, .0715, .0836, .0876, .0878, .0889, .0903, .0914, .0916, .0923, .0927–.0929, .0947, .0949, .0957, .0973, .0976, .0980, .0981, .0985, .0989, .167, .180, .181, .195, .346, .368, .369, .395, .453, .458, .516, .520, .529, .579, .582, .583, .602, .617, .633, .635, .637, .667, .740, .773, .986 and .994.

B

$x = .21$, *for* .21156, *read* .21157; $x = .39$, *for* .39063, *read* .40063; $x = .47$, *for* .48918, *read* .48929.

Although in 1932 Hayashi issued a pamphlet of corrections to his *Sieben- und mehrstellige Tafeln* he failed to note any errors in his $\sin^{-1} x$ values.

H. E. Salzer and E. Isaacson

Mathematical Tables Project
New York City

Editorial Notes.—It seems desirable to elaborate the incidental reference to the 12-page pamphlet in which Hayashi published an extensive list of corrections. It is entitled *Berichtigung in Hayashis sieben* [*sic*] *u. mehrstellige Tafeln* (1926), Fukuoka, 1932. 19.1 x 26.4 cm. The photoprint of this pamphlet in the Library of Brown University was made from an original in the Library of Columbia University. There are 8 pages (with 3 columns on a page) of corrections; in e^x and e^{-x}, $1\frac{1}{2}$ cols. (about 75 corrections); in $\sin x$ and $\cos x$, about 11 cols. (550 corrections), etc. Eleven corrections of the *Berichtigung*, and six additional corrections have been made by hand (presumably Hayashi's). L.J.C. writes as follows, however: "This pamphlet has the same percentage of error as the original volume and, what is more, there are some cases where things that were right in the original are put wrong by these so-called corrections."

We may also add notes regarding errors in radian tables of $\tan^{-1} x$ in the above mentioned works of Hayashi, *A* and *B*. In *A* the tables are for $x = [0.00000(0.00001)0.001; 20D]$, $[0.0000(0.0001)0.0999; 10D]$, $[0.000(0.001)2.999; 8D]$, $[3.00(0.01)10.0(0.1)20(1)50; 7D]$. In his *Tables of* $\tan^{-1} x$ *and* $\log(1+x^2)$, (*Tracts for Computers*, no. XXIII, 1938), L. J. Comrie states that Hayashi's tables were first checked through the first quadrant, i.e. up to $x = 1.570$, and 100 errors exclusive of errors in the last decimal were found. Comrie remarked

further that over 100 errors were found for later values of x. In the above mentioned Hayashi pamphlet 71 corrections are given for these tables. In B, 13 errors not mentioned in his errata occur in the table for $\tan^{-1} x$, $x = [0(0.01)10.00; 5D]$, and are listed in *The Table of Arc Tan x* by the MATHEMATICAL TABLES PROJECT (New York, 1942); see RMT **90**.

36. P. R. E. JAHNKE and F. EMDE, *Table of Functions*, first–fifth eds., 1909–1943. Compare RMT **113**; MTE **21**, **23**, **33**, **34**, **37**. Page references are to the first and fifth editions.

P. 54 (62), $F(15°, 32°)$, *for* 0.5604, *read* 0.5603. P. 56, $F(60°, 5°)$, *for* 0.08745, *read* 0.08735, error in first ed. only. P. 61 (69), $E(15°, 16°)$, *for* 0.2788, *read* 0.2790; $E(20°, 16°)$, *for* 0.2786, *read* 0.2788; $E(25°, 16°)$, *for* 0.2790, *read* 0.2786. P. 64 (72), $E(50°, 64°)$, *for* 1.0072, *read* 1.0007; $E(80°, 64°)$, *for* 0.9027, *read* 0.9072; $E(90°, 86°)$, *for* 0.9926, *read* 0.9976. P. 68 (85), $E(8°)$, *for* 1.5630, *read* 1.5632, error in first and second eds. only; $K(86°48')$, *for* 4.2744, *read* 4.2746; $K(87°36')$, *for* 4.5619, *read* 4.5609; $K(89°36')$ *for* 6.3504, *read* 6.3509. Except in the two cases noted these errors occur in all five editions.

S. P. GLAZENAP, *Matematicheskie i Astronomicheskie Tablit͡sy*, Leningrad, 1932, p. 214–215.

37. P. R. E. JAHNKE and F. EMDE, *Table of Functions*, third, fourth and fifth eds., 1938–1943. Compare RMT **113**, and MTE **36**.

On p. 79 the following recurrence formula is given for the complete Legendre elliptic integral:

$$4(n+1)^2\int \mathbf{E}x^n dx - (2n+3)(2n+5)\int \mathbf{E}x^{n+1}dx = 2x^{n+1}[\mathbf{E} - (2n+3)(1-x)\mathbf{K}].$$

It is readily verifiable that the right hand member should be

$$2x^{n+1}\{[2n+1-(2n+3)x]\mathbf{E} + (1-x)\mathbf{K}\}.$$

I. OPATOWSKI

Illinois Institute of Technology

EDITORIAL NOTE.—This formula was given correctly in the source indicated by EMDE, namely: K. F. MÜLLER, *Archiv f. Elektrotechnik*, v. 17, 1926, p. 337.

38. MATHEMATICAL TABLES PROJECT, *Tables of Arc Tan x*, New York, 1942. Compare RMT **90**.

P. 27, $x = 2.554$, 2.557–2.559, insert missing decimal points in function.
P. 27, $x = 2.570$, insert missing decimal point in argument.
P. 41, $x = 3.999$, insert missing decimal point in function.
P. 68, $x = 6.645$, insert missing 1 in function.
P. 160 (some volumes, not all) $x = 5450$, insert missing 1 in function.

MRS. RUTH CAPUANO

50 Church St., New York City

39. MATHEMATICAL TABLES PROJECT, *Tables of the Exponential Function e^x*, New York, 1939.

The integral part of the entry corresponding to the argument 2.3979 on p. 240 should be 11 in lieu of 10.

A. N. LOWAN

40. MATHEMATICAL TABLES PROJECT, *Table of Natural Logarithms*, 1941.

V. 1, p. 82, argument, *for* 9023, *read* 8023; v. 2, p. 355, argument, *for* 85201, *read* 85301.

J. W. WRENCH, Jr.

41. A. J. Thompson, *Tables of the Coefficients of Everett's Central-Difference Interpolation Formula*, London, second ed., 1943. See RMT **148**.

An error in the first edition (1921) was pointed out by Chappell (see MTE **31**).

Page 10, ϵ_2 for $\phi = 0.468$, *for* .06091 51280, *read* .06091 61280. A recent comparison by Mr. E. S. Davis has shown that there are no other discrepancies between the two editions. The columns ϵ_2 and ϵ_4 have been checked by Miss D. P. Kilner by comparison with the appropriate columns of my Burroughs-script tables of 4-point and 6-point Lagrangean coefficients. From these checks, and Thompson's known high standard (no errors are known in his monumental *Logarithmetica Britannica*), there is every reason to believe that the rest of his figures are correct.

The following trivial corrections in the text have been pointed out to me in letters from Thompson and J. C. P. Miller.

Page	line	*for*	*read*	authority
viii	12	$\delta^8_{1.0}$	$\delta^8u_{1.0}$	J.C.P.M.
32	-2	δ^2 to δ^{10}	ϵ_2 to ϵ_{10}	A.J.T.

L. J. C.

42. H. Weber, *Theorie der Abelschen Functionen vom Geschlecht* 3, Berlin, 1876. The following corrections in Table II, p. 183 (The complete system of odd characteristics), were given in a letter, dated 10 February 1933, from the late H. S. White to the late W. F. Osgood.

"Left-hand column, 7th characteristic from the top, instead of $p\binom{000}{011}, \cdots,$ it should read (at least this is one of the 8 correct systems)

p	β_1	β_2	β_3	β_4	β_5	β_6	β_7
$\binom{000}{111}$	$\binom{111}{111}$	$\binom{001}{011}$	$\binom{110}{100}$	$\binom{101}{011}$	$\binom{010}{111}$	$\binom{100}{100}$	$\binom{011}{110}$

Otherwise, replace this characteristic, no. 7, left, by the no. 11, right, inverted, with its 7β's also inverted.

"In system 9, right-hand column, replace β_4 by $\frac{100}{110}$.

In system 8, right-hand column, replace β_3 by $\frac{111}{100}$.

In system 12, right-hand column, replace β_4 by $\frac{011}{101}$.

In system 13, right-hand column, replace β_4 by $\frac{100}{101}$.

In system 11, left-hand column, replace β_4 by $\frac{010}{111}$.

In system 17, right-hand column, replace β_4 by $\frac{100}{101}$.

It is singular that Weber should have let so many errors slip by! And notice that five of them are in column β_4. That must indicate something about his field of vision."

43. H. E. H. Wrinch and D. M. Wrinch, *Phil. Mag.*, s. 6, v. 47, 1924, p. 63, $I_0(30) = 7.81674 \times 10^{11}$; R. C. Colwell and H. C. Hardy, *Phil. Mag.*, s. 7, v. 24, 1937, p. 1046, $I_0(30) = 5.70 \times 10^{11}$.

The Editors brought these contradictory results in tables, already referred to in *MTAC* (p. 139–140), to my attention, and suggested that it would be a matter of interest to have the correct result determined.

In the asymptotic expansion

$$I_0(x) = \frac{e^x}{(2\pi x)^{1/2}}\left\{1 + \frac{1^2}{1!8x} + \frac{1^2\cdot 3^2}{2!(8x)^2} + \frac{1^2\cdot 3^2\cdot 5^2}{3!(8x)^3} + \cdots\right\},$$

upon setting $x = 30$, and evaluating 25 terms of the series, each to 22 decimal places, I found the sum of the terms in the braces to be

$$1.00424\ 76530\ 20713\ 59155(8).$$

This calculation was performed twice,—first by means of a calculating machine, the second time long-hand. The results agreed perfectly.

The value of e^{30} was found in the W.P.A. *Tables of the Exponential Function* e^x (1939), p. 533, to 19S. Subsequently I checked and extended this approximation to 38S. I have unpublished values of $\pi^{1/2}$ and $1/\pi^{1/2}$ to 317 and 310 decimal places respectively. These were calculated from π and $1/\pi$, respectively, and the product of the square roots was formed with the assistance of a machine, and was found to consist of a sequence of 309 consecutive 9's. Consequently I have great confidence in the accuracy of these roots. The value of $15^{1/2}$ was determined to 40D. By multiplication I found $e^{30}15^{1/2}/30\pi^{1/2} = 7.78366\,06884\,04464\,04193\,55906\,75 \times 10^{11}$; and finally, $I_0(30) = 7.81672\,29782\,39774\,8972 \times 10^{11}$, correct to 20S.

Hence the value of Wrinch and Wrinch is nearly correct, the error being in the last significant figure which they give. The value of Colwell and Hardy is entirely incorrect.

J. W. WRENCH, Jr.

4211 Second St., N.W.
Washington 16, D. C.

UNPUBLISHED MATHEMATICAL TABLES

References have been made to Unpublished Mathematical Tables in RMT **138** (DAVIS, MILLS), and RMT **157** (DAVIS, MATH. TABLES PROJECT).

21[B].—*Tables of Fractional Powers*, Mss. prepared by, and in possession of, the MATHEMATICAL TABLES PROJECT, 50 Church St., New York City.

Of these 12 tables 6 are of A^x, and 6 of x^a, as follows:

I: $A = 2(1)9$, and $x = [0.001(0.001)0.01(0.01)0.99$; 15D].
II–IV: $A = 10$, π, and Euler constant, $x = [0.001(.001)1.000$; 15D].
V: $A = 0.01(0.01)0.99$, and $x = [0.001(0.001)0.01(0.01)0.99$; 15D].
VI: $A = p10^{-3}$, $p =$ the primes between 101 and 997, and $x = [0.001(0.001)0.01$-$(0.01)0.99$; 15D].
VII–XI: $x = [0.01(0.01)9.99$; 15D] and $a = \pm\frac{1}{4}, \pm\frac{1}{3}, \pm\frac{3}{4}, \pm\frac{1}{5}, \pm\frac{2}{3}$.
XII: $a = 0.01(0.01)0.99$. For any a the values of x^a were computed at the interval 0.001 in x approximately up to that value of x for which the derivative of the function (which is ∞ at $x = 0$) is in the neighbourhood of unity. All entries are to 7D.

A. N. LOWAN

22[D, H].—*Roots of the equation* $\tan x + ax = 0$, $a = 3/(8\pi)$, $3/(12\pi)$, $3/(16\pi)$. Ms. in possession of the Department of Aeronautical Engineering, University of Michigan, Ann Arbor, Mich. Compare Q. **8**.

The first three roots of each of these equations were calculated to 5S, and the next five roots to 3S. First, we obtained an approximate value x_n^0, graphically, and then applied

a correction

$$\epsilon_n = -(\tan x_n^0 + a x_n^0)/(\sec^2 x_n^0 + a).$$

The corrected values $x_n^0 + \epsilon_n$ are about as accurate as we need them.

OTTO LAPORTE

University of Michigan

23[L].—HANSRAJ GUPTA, *Table of Liouville's function and its sum*. Ms. in possession of the author, Government College, Hoshiarpur, Punjab, India.

Liouville's function $\lambda(n)$ may be defined for the positive integer argument $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ to be $+1$ or -1 according as $m = \alpha_1 + \alpha_2 + \cdots + \alpha_t$ is even or odd. This table gives $\lambda(n)$ and also the sum

$$L(n) = \lambda(1) + \cdots + \lambda(n)$$

for $n \leqslant 20\,000$. The function $L(n)$ is clearly the excess of the number of those integers n which have an even number of prime factors over the number of integers having an odd number of prime factors. The reason for computing this table is to verify or refute an important conjecture of PÓLYA (Deutsche Math.-Ver. *Jahresb.*, v. 28, 1919, p. 38) to the effect that this "excess" is really negative or zero for $n > 1$. The table shows that the conjecture is correct as far as $n = 20\,000$ but does not show that the conjecture has safely passed its worst trials. In fact one finds that $L(48\,512) = -2$. Data, taken from this table, on the behavior of $L(n)$ have been published by the author, Indian Acad. Sci., *Proc.*, v. 12A, 1940, p. 407–409. Current interest in Pólya's conjecture (which implies the Riemann Hypothesis) has been heightened by the recent results of A. E. INGHAM, *Amer. J. Math.*, v. 64, 1942, p. 313–319. The corresponding problem for the companion function $\mu(n)$ of MÖBIUS and its sum is also unsolved and has been the subject of much tabulation; see *Guide to Tables in the Theory of Numbers*, Nat. Res. Council, *Bull.* no. 105, Washington, D. C., 1941, p. 9.

D. H. L.

NOTES

17. DESIRABLE MATHEMATICAL TABLES FOR REPUBLICATION.—The Office of Alien Property Custodian has licensed, during the past several months, the reprinting of scientific and technical books, of enemy origin, which are not available in a quantity sufficient to meet the demands of the wartime operations of science and industry. In RMT **79**, **113**, **128**, references have been made to three volumes of this kind. Before definite decision can be made regarding the licensing of additional Mathematical Tables for republication, it is necessary for the Custodian to be informed about the extent of the need of such tables and to receive suggestions of specific titles for consideration. This can be accomplished if suggestions of specific significant tables, or any inquiries, are sent by individuals to the undersigned at Division of Patent Administration, Office of Alien Property Custodian, Washington, D. C.

H. H. SARGEANT, Chief

18. *Phil. Mag.* TABLES, SUPPL. 1 (see *MTAC*, p. 135–141).—In H. C. PLUMMER, "The numerical solution of a type of equation," s. 7, v. 32, 1941, p. 505–512, roots are found of the following transcendental equations, among others, of the type $\tan x = xf(x)$, where $f(x)$ is a single-valued function:

I. For $\tan x = x/(1 - x^2)$, which relates to electrical waves on a sphere (J. J. THOMSON, *Notes on Recent Researches in Electricity and Magnetism*, Oxford, 1893, p. 373), are given the first 5 positive roots to 4D. The first 3 of these roots agree with those of L. SILBERSTEIN, *Bell's Mathematical Tables*, London, Bell, 1922, p. 97. Silberstein's fourth root is appreciably in error. Before either of these results had been published, H. LAMB gave the first 6 positive roots to 4D in London Math. So., *Proc.*, v. 13, p. 202, 1882.

II. $\tan x = 2x/(2 - x^2)$, an equation found in connection with discussion of sound waves, or the free oscillations of gas, in a rigid spherical shell (RAYLEIGH, *The Theory of Sound*, v. 2, 1878, p. 232; second ed., 1896, p. 265). Five positive roots to 4D are given. For this equation also LAMB gave (*l.c.*) 6 positive roots to 4D.

III. Another equation considered by Rayleigh (*ibid.*, v. 2, 1896, p. 266) is $\tan x = x(x^2 - 9)/(4x^2 - 9)$ for which he gives one non-zero solution to 4D; Plummer finds four positive roots to 4D.

IV. The equations $\cos x \cdot \cosh x = \pm 1$, occur in connection with the lateral vibration of bars (RAYLEIGH, *ibid.*, v. 1, 1877, p. 222–224; second ed., 1896, p. 278). Four positive solutions, to 6D, are found for each of these equations. In this connection Rayleigh's errors in the first edition are corrected in the second. The oversight was noted by A. G. GREENHILL (*Mess. Math.*, v. 16, 1886, p. 119) who has given another interesting application of the same equations. More roots and general formulae are given in E. P. ADAMS, *Smithsonian Mathematical Formulae*, Washington, D. C., 1939 (1922), p. 86.

H. B. and R. C. A.

19. *Phil. Mag.* TABLES, SUPPL. 2.—In H. CARRINGTON, "The frequencies of vibration of flat circular plates fixed at the circumference," s. 6, v. 50, 1925, p. 1261–1262, the 16 roots, < 16, of $J_{n+1}(x) \cdot I_n(x) + I_{n+1}(x) \cdot J_n(x) = 0$, for $n = 0, 1, 2, 3$, are given to 5S. The first ten roots for each of these values of n are given by J. R. AIREY, "The vibrations of circular plates and their relation to Bessel functions," Phys. So. London, *Proc.*, v. 23, 1911, p. 227; for $n = 0$ these roots are given to 4D, but for the others, to 3D. Carrington's first root for $n = 0$ was 3.1961; according to Airey this should be 3.1955. Of the 10 roots given in JAHNKE and EMDE (1933), p. 283, and 1938–43 editions, p. 234, 8 are erroneous to the extent of 1 to 20 points in the last decimal places.

R. C. A.

20. A ROOT OF THE EQUATION $10 \log x = x$.—In *Mathematical Gazette*, v. 15, p. 367, 1931, C. V. BOYS gives what purports to be a 60D approximation to the irrational root of this equation. I checked this value and found it correct to only 38D. By Newton's method I deduced a value of x to 66D and then in a second calculation determined the logarithm of this last approximation. It developed that my result was correct to 65D. Incidentally I employed the rational approximation $x = 2^7.5.7/3^3.11^2$. The correct value for x, to 65D, is as follows:

1.37128 85742 38623 53686 13621 06299 68995 88428 54404 84225-
70704 08723 85385.

J. W. WRENCH, JR.

Editorial Note.—Part of this value is the first in a table of ten "coincidental logarithms" listed by D. M. Knappen,[1] solutions (8 of them irrational) of the equations $10^n \log x = x$, $n = 1, 2, \cdots, 10$. To 50D it agrees with the result of Mr. Wrench; in the remaining three places there is disagreement, 650 instead of 707. For $n = 8$ the value given for x, to 66D, is as follows:

$$\begin{array}{l} 8.95191\ 59982\ 67846\ 23159\ 99873\ 77688\ 49072\text{-} \\ \quad 72932\ 33707\ 59495\ 47848\ 84326\ 51391\ 4 \qquad \times 10^8. \end{array}$$

[Upon bringing this result to the attention of Mr. Wrench he verified its accuracy through 64D and then added the following 16 digits: 0 73919 87374 90256.] Knappen states that these remarkable numbers were called "constants" by Oliver Byrne, who certainly discussed such numbers in at least two publications,[2] of 1849 and 1864. In each of these places Byrne considered the problem "to find a number whose common logarithm is composed of the same digits and in the same order as itself." In the first he gives the ten values, each to 15D, but the last two or three figures of eight of these are incorrect. Similar remarks apply to the seven values given in the second publication, where Byrne recklessly writes "no known development but the dual will establish" them. As yet we have not been able to connect with Byrne, the 53 to 66D values given by Knappen. Perhaps some reader can supply this information.

[1] D. M. Knappen, *Math. Mag.*, Washington, D. C., v. 1, p. 202—the number containing this page is dated Oct. 1884, although published in August 1887.

[2] O. Byrne, (a) *Practical, Short, and Direct Method of Calculating the Logarithm of any given Number, and the Number corresponding to any given Logarithm.* London and New York, 1849, p. viii, 14–15, 52, 65–76. Compare J. Henderson, *Bibliotheca Tabularum Mathematicarum*, Cambridge, 1926, p. 121–122.

(b) *Dual Arithmetic. A New Art. . . . New issue with a Complete Analysis*, London, 1864, *Analysis*, p. 75–78.

QUERIES

8. Roots of the equation $\tan x = cx$.—As early as 1748 in v. 2 of his *Introductio in Analysin Infinitorum*, Lausanne, p. 318–320, Euler solved the problem "Invenire omnes Arcus, qui Tangentibus suis sint aequales," by deriving a general formula, from which he found the first 10 non-negative roots of $\tan x = x (c = 1)$. The second of these roots was given in the form "$3.90° - 12° 32' 48''$" [4.49340834 instead of 4.49340964], and the tenth as "$19.90° - 1° 55' 16''$." These results may be regarded as correct to 5D. $J_{3/2}(x) = 0$ is equivalent to Euler's equation $\tan x = x$. In 1886 Lommel listed the first 17 non-negative roots, to 6D (Akad. d. Wiss., Munich, *Math.-naturw. Abt.*, *Abhandlungen*, v. 15, p. 651). This list, abbreviated to 4D, was published in the first, second and fifth editions of Jahnke and Emde (see RMT **113**), and in E. P. Adams, *Smithsonian Mathematical Formulae* . . . , Washington, D. C., 1939 (1922), p. 84. The first 36 roots, to 4D, are given in K. Hayashi, *Fünfstellige Funktionentafeln*, Berlin, 1930, p. 52. The 18th to the 36th roots, to 6D, are given in L. Silberstein, *Bell's Mathematical Tables*, London, Bell, 1922 (= *Synopsis of Applicable Mathematics with Tables*, New York, Van Nostrand, 1923), p. 97. In 1827 Cauchy found the first 4 roots to 7D when $c = 1$, the first 5, to 6D, when $c = 4$, and the first 4, to 4–6D, when $c = 8/5$ ("Théorie de la propagation des ondes à la surface d'un fluide pesant d'une profondeur indéfinie," Acad. d. Sci., Paris, *Mémoires prés. par divers Savants*, s. 2, v. 1, p. 198–209; *Oeuvres*, s. 1, v. 1, Paris, 1882, p. 204–212). Compare UMT **22**. What other solutions of $\tan x = cx$ are known?

H. B. and R. C. A.

9. System of Linear Equations.—Where may one find an adequate treatment of the method of Gauss and Seidel for solving a system of n

linear equations in n unknowns by successive approximations? The discussion given in WHITTAKER and ROBINSON, *The Calculus of Observations* (London, 1924, and third ed., 1940, p. 255–256), is not satisfactory. The part purporting to show that the process always improves a trial solution suffers the following simple exception:

$$2x + y = 1, \quad x + 3y = -1.$$

Here the initial solution $x = 1/2, y = -1/3$ is not improved by replacing x by $2/3$ as required by the process.

D. H. L.

QUERIES—REPLIES

8. TABLES OF $N^{3/2}$(Q5, p. 131).—Another table for three-halves powers of numbers to more than three places is T. 70, p. 290 of J. T. FANNING, *A Practical Treatise on Hydraulic and Water-Supply Emergency*, tenth ed., New York, 1892, where N = [0.04(0.01)0.20(0.02)1.0(0.1)4; 4D].

H. B.

CORRIGENDA

P. 2, l. 31, *for* Reply to Query **6**, *read* Reply-to-Query **6**. P. 6, l. 6, *for* v. 4, *read* v. 14. P. 9, **76** *for* CHAPMAN, *read* CHAPIN. P. 14, l. 5 from bottom, *for* 0.001, *read* 0.0001. P. 15, l. 6, *add* Also, p. 224–224c, sin x, cos x to 10D, log sin x, log cos x to 5D, x = 0(.1)10, 0(1)100. P. 15–16, *omit* references to HAYASHI tables of sin $\frac{1}{2}x\pi$, cos $\frac{1}{2}x\pi$, l. 13–14 from bottom of p. 15; also to tables of KOLKMEIJER and BUERGER, top of p. 16. P. 16, l. 8 from bottom, *for* Spoon, *read* Spon. P. 18. l. 1 and 2 from bottom, *for* 6D, *read* 6D-7D. P. 19, l. 3 from bottom, *for* x, ⋯3D], *read* x = [0.00(0.01)1.0(0.1)10(1)-100(10)1000; 3D]. P. 20, footnote, l. 6, after "109." *insert* With the aid of the entries presented the logarithms of all numbers N = 1(1)109 are readily found. P. 47, **90**, l. 3, *for* State, *read* City. P. 69, **2**, l. 3, *for* Houghton, *read* Haughton; **3**, l. 1, *for* 12S, *read* 10S–12S; **5** and **6**, *for* with differences, *read* with first differences.

P. 70, **8**, l. 2, *for* 10D, *read* 9D–10D; l. 4, *for* 0(1/2)(13/2), *read* 0($\frac{1}{2}$)6$\frac{1}{2}$; l. 5, *for* $\frac{1}{4}n\pi$ *read* $\frac{1}{2}n\pi$, [this was a mistake in the *Report*]; **10**, l. 3, *for* by degrees, *read* at three-degree intervals; **12**, l. 3, *for* 80°1, *read* 80°.1. P. 73, **44**, l. 2, Ei in roman, not ital.; **49**, l. 4, *for* 0.0(0.1)10.0 *read* 0.0(0.1)7(1)10. P. 74, **52**, l. 20, *for* J_k^0 and J_k^1, *read* I_k^0 and I_k^1; **56**, l. 4, *for* 120, *read* 12.0. P. 96, in UMT **9**, totals, make the following changes: 390 *for* 391; Poulet 65 (*for* 68); Escott 233 (*for* 235); and *add* Poulet and Gérardin 4 (1929). P. 109, l. 17–18, *for* $J_1(17)$, *read* $J_1(x_{17})$; l. 20–22, for these lines *read*, the roots of $J_1(x)N_1(kx) - J_1(kx)N_1(x) = 0$ on p. 204 of nos. **3–5**, p. 274 of no. **2**, and p. 162 of no. **1**, the first three roots for the value $k = 2$ should be 3.1917, 6.3116, and 9.4446 according to values given in MUSKAT, ⋯. P. 108, l. 17, *for* Debeye, *read* Debye.

P. 125, l. 20–23, *for* numbers, *read* figures. P. 138, **26**, l. 4, *for* $J_{+\frac{1}{3}}(x)$, *read* $J_{\pm\frac{1}{3}}(x)$; *for* uncertain fourth, *read* approximate fifth; l. 5, *for* $J_{+\frac{1}{3}}(x)$, *read* $J_{\pm\frac{1}{3}}(x)$; l. 5–6, for uncertainties, *read* approximate fifths; l. 7, *for* $\frac{1}{4}(n + 1)$, *read* $\frac{1}{4}/(n + 1)$. P. 140, no. **38**, *for* ∂x, and ∂x^2, *read* ∂v and ∂v^2. P. 143, l. 4 from bottom, *for* einen, *read* einem. P. 145, for line 8, *read:* place tables for A with D = 0.0000(0.0001)2.000(0.001)4.00(0.01)6.94; and for S with D = 0.3000(0.0001)2.000(0.001)4.00(0.01)6.94. P. 157, l. 16–17, *for* $B_n^{(n)}(0)$ and $B_n^{(n)}(1)$, *read* $B_n^{(n)}(0)/n!$ and $B_n^{(n)}(1)/n!$ P. 161, l. 11, *delete* "P. 54, F(35°, 30°), *for* 0.6220, *read* 6200." P. 161, l. 13, *for* 1035, *read* 1037. P. 164, l. 11 from bottom, eliminate the second "10;". P. 168, l. 26, *for* Küster, *read* Küstner. P. 169, l. 27, *read Physical;* l. 6 from bottom, *for kkadratov, read kvadratov;* l. 4 from bottom, *read* Izdatyel'stvo.

is-
ons
The
on

ng

ers
A
d.,

14.
001.
5D,
$x\pi$,
16.
6D,
(1)-
ries
. 3,
ead

ead
gree
. 4,
I_k^1;
390
29).
s of
no.
ord-

cer-
ties,
and
e 8,
or S
and
for
nate
cal;
o.

Cl

A
B
C
D
E

F

G

J

th
in

W

CLASSIFICATION OF TABLES, AND SUBCOMMITTEES

A. Arithmetical Tables. Mathematical Constants
B. Powers
C. Logarithms
D. Circular Functions
E. Hyperbolic and Exponential Functions
  Professor DAVIS, *chairman*, Professor ELDER
  Professor KETCHUM, Professor LOWAN

F. Theory of Numbers
  Professor LEHMER

G. Higher Algebra
  Professor LEHMER
H. Numerical Solution of Equations
J. Summation of Series

I. Finite Differences. Interpolation
K. Statistics
  Professor WILKS, *chairman*, Professor COCHRAN, Professor CRAIG
  Professor EISENHART, Doctor SHEWHART

L. Higher Mathematical Functions
M. Integrals
  Professor BATEMAN

N. Interest and Investment
O. Actuarial Science
  Mister ELSTON, *chairman*, Mister THOMPSON, Mister WILLIAMSON

P. Engineering

Q. Astronomy
  Doctor ECKERT, *chairman*, Doctor GOLDBERG, Miss KRAMPE

R. Geodesy
S. Physics
T. Chemistry
U. Navigation

Z. Calculating Machines and Mechanical Computation
  Doctor COMRIE, *chairman*, Professor CALDWELL, *vice-chairman*
  Professor LEHMER, Doctor MILLER, Doctor STIBITZ, Professor TRAVIS

EDITORIAL NOTICES

The addresses of all contributors to each issue of *MTAC* are given in that issue, those of the Committee being on cover 2. The use of initials only indicates a member of the executive committee.

The next number of *MTAC* will be a greatly enlarged special issue wholly devoted to a *Guide to Tables in Bessel Functions*.

Volume I is to contain at least 8 numbers continuously paged.

CONTENTS

APRIL 1944

LANCASTER PRESS, INC., LANCASTER, PA.

6

3

0

1

3

4

4

4